Definition: The **rank** of a matrix is the number of nonzero rows in its REF or RREF.

solvable

Fact: If a system is consistent then:
rank+(# of parameters in solution)=# of variables

Example: Verify the fact for the following system:

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

RREF

$z = t$

rank = 2
# of parameters = 1
# of variables = 3

2 + 1 = 3 ✓

Example: Rephrase the fact in terms of columns of the coefficient matrix.

(# of columns with pivots)
+ (# of columns without pivots)
= # of columns

$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

Comment: Notice that $\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$ is a solution to the following system:

$$\begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 0 \end{aligned}$$

Definition: A system whose constants are all zero is called a **homogeneous system**. The solution $\vec{x} = \vec{0}$ is called the **trivial solution**.

Fact: A homogeneous system always has at least one solution: $\vec{x} = \vec{0}$.

Example: Consider a homogeneous system with more variables than equations. How many solutions does the system have?

$$m\left\{\left[\begin{array}{c|c} \underbrace{\qquad\qquad\qquad\qquad}_{n} & \begin{matrix}0\\0\\\vdots\\0\end{matrix}\end{array}\right]\right. \qquad (n>m)$$

System has at least 1 solution: $\vec{x}=\vec{0}$.

There will be at least 1 parameter in the solution:

$$\left[\begin{array}{ccccc|c} 1 & & & & & 0\\ & 1 & & & & 0\\ & & 1 & \ddots & & \vdots\\ & & & & 1 \quad & 0 \end{array}\right] \text{ REF}$$

$\Rightarrow$ System has infinitely-many solutions.

2.3 Span and Linear Independence

Example: Is $\begin{bmatrix} 8 \\ -10 \end{bmatrix}$ a linear combination of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

Let $c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$

$$\begin{bmatrix} -c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ -3c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$$

$$\begin{cases} -c_1 + 2c_2 = 8 \\ 2c_1 - 3c_2 = -10 \end{cases}$$

$$\begin{array}{ccc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 2 & 8 \\ 2 & -3 & -10 \end{array}\right]$$

$$\xrightarrow{\frac{R_1}{-1}} \left[\begin{array}{cc|c} 1 & -2 & -8 \\ 2 & -3 & -10 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & -8 \\ 0 & 1 & 6 \end{array}\right] \text{ REF}$$

System is consistent (solvable) $\Rightarrow$ YES

Optional:

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right] \text{ RREF}$$

$c_1 = 4$, $c_2 = 6$

$$4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix} \checkmark$$

Example: Is $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ a linear combination of $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$?

Let $c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

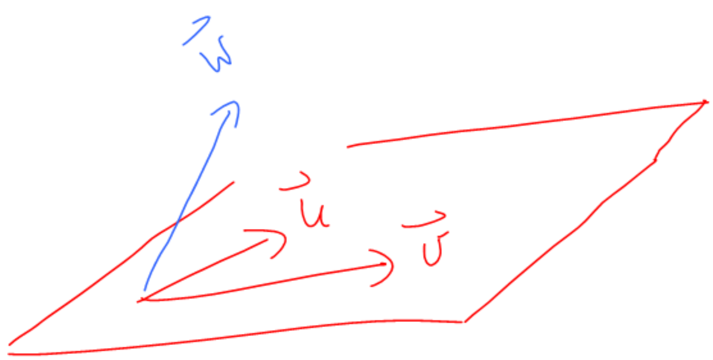
$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{array}\right] \end{array}$$

$R_3 - R_1$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{array}\right]$

$\frac{R_2}{3}$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & -1 & 1 \end{array}\right]$

$R_3 + R_2$ $\quad \left[\begin{array}{cc|c} & & \\ & & \\ 0 & 0 & \frac{4}{3} \end{array}\right]$ REF

System is inconsistent (unsolvable) $\Rightarrow$ NO



exactly when

Fact: The vector $\vec{b}$ is a linear combination of the columns of matrix A if and only if the system $\left[A|\,\vec{b}\right]$ is consistent.

Definition: The **span** of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is the set of all linear combinations of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

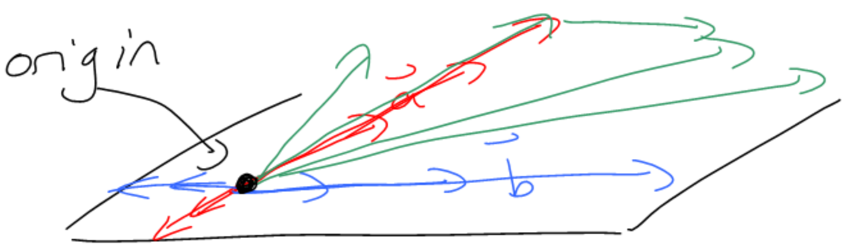
Comment: a) span$(\vec{a}, \vec{b})$= $\{\vec{0},\ 3\vec{a},\ -7\vec{b},\ 2\vec{a}+5\vec{b}, \ldots\}$

b) span$(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$= $\{c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n\}$
where $c_1, c_2, \ldots, c_n$ are any real numbers.

Fact: The zero vector $\vec{0}$ is in span$(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ because $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_n = \vec{0}$.

Suppose $\vec{a}$ and $\vec{b}$ are not parallel.

span$(\vec{a}, \vec{b})$ is a plane:



A plane can be viewed as a set of vectors, or a set of points (by unvectorizing).

Algebra Language: span$(\vec{a}, \vec{b}, \vec{c})$ contains $\vec{0}$

Geometry Language: span$(\vec{a}, \vec{b}, \vec{c})$ contains the origin