Assignment 1 is on website

Due Thurs Sept 18 at 11:30 am

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1.4 The Cross Product

The cross product $\vec{u} \times \vec{v}$ is defined for $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$.

Example: Let $\vec{u} = [1, 2, 1]$ and $\vec{v} = [3, -1, 4]$. Calculate $\vec{u} \times \vec{v}$.

$$\begin{array}{ccccccccc} 1 & & 2 & & 1 & & 1 & & 2 \\ & & & \times & & \times & & \times & \\ 3 & & -1 & & 4 & & 3 & & -1 \end{array}$$

$$\vec{u} \times \vec{v} = [2(4) - 1(-1), 1(3) - 1(4), 1(-1) - 2(3)]$$

$$= [9, -1, -7]$$

Example: Let $\vec{u} = [1, 2, 1]$ and $\vec{v} = [3, -1, 4]$. Calculate:
a) $\vec{v} \times \vec{u}$

$$\begin{array}{ccccccccc} 3 & & -1 & & 4 & & 3 & & -1 \\ & & & \times & & \times & & \times & \\ 1 & & 2 & & 1 & & 1 & & 2 \end{array}$$

$$\vec{v} \times \vec{u} = [-9, 1, 7]$$

b) $(\vec{u} \times \vec{v}) \cdot \vec{u}$

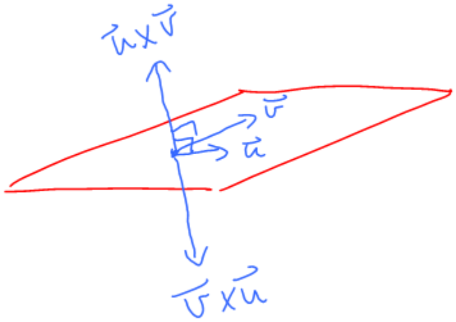
$$[9, -1, -7] \cdot [1, 2, 1]$$

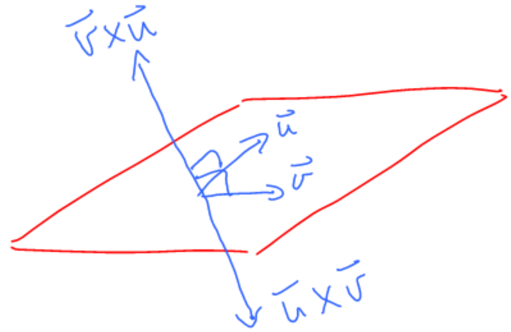
$$= 9(1) + (-1)(2) + (-7)(1)$$

$$= 0$$

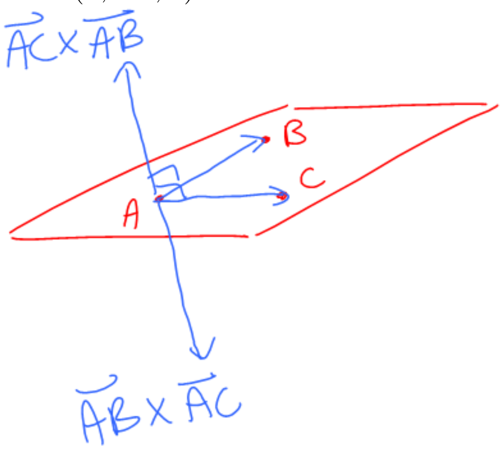
Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Then:
$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ AND
$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

Fact: The vector $\vec{u} \times \vec{v}$ is a normal for the plane containing $\vec{u}$ and $\vec{v}$. The direction of $\vec{u} \times \vec{v}$ is determined by the Right Hand Rule.





Example: Find the general form of the plane through $A = (1, 3, 6)$, $B = (2, 1, 4)$ and $C = (1, -1, 5)$.



$$\vec{AB} = [1, -2, -2]$$
$$\vec{AC} = [0, -4, -1]$$
$$\vec{n} = \vec{AB} \times \vec{AC}$$
$$= [-6, 1, -4]$$

$$\boxed{\begin{array}{ccccc} 1 & -2 & -2 & 1 & -2 \\ 0 & -4 & -1 & 0 & -4 \end{array}}$$

Normal form $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

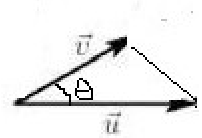
$$\begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

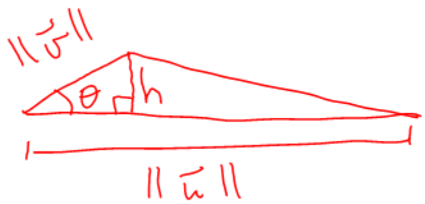
General Form $\quad -6x + y - 4z = -27$

Comment: Recall that $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.

Fact: If $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^3$ then $||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}|| \sin\theta$.

Example: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Consider the triangle below.
Show that the area of the triangle is $\frac{1}{2}||\vec{u} \times \vec{v}||$



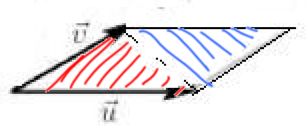


$$\sin\theta = \frac{h}{||\vec{v}||}$$

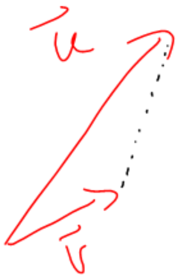
$$||\vec{v}|| \sin\theta = h$$

$$A(\Delta) = \frac{1}{2} \text{ base} \cdot \text{height}$$
$$= \frac{1}{2} ||\vec{u}||\ ||\vec{v}|| \sin\theta$$
$$= \frac{1}{2} ||\vec{u} \times \vec{v}||$$

Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Consider the parallelogram below, which can be divided into two triangles with equal area. Then:
Area(triangle)$= \frac{1}{2}||\vec{u} \times \vec{v}||$ AND
Area(parallelogram)$= ||\vec{u} \times \vec{v}||$



Example: Find the area of the triangle determined by $\vec{u} = [1, 4, 5]$ and $\vec{v} = [2, 3, 6]$.



$$\vec{u} \times \vec{v} = [9, 4, -5]$$

$$\boxed{\begin{array}{ccccc} 1 & 4 & 5 & 1 & 4 \\ 2 & 3 & 6 & 2 & 3 \end{array}}$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{9^2 + 4^2 + (-5)^2}$$

$$= \sqrt{122}$$

$$\text{Area}(\Delta) = \frac{\|\vec{u} \times \vec{v}\|}{2}$$

$$= \frac{\sqrt{122}}{2}$$

Definition: A **matrix** is a rectangular array of numbers. For example, $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Definition: The **determinant** of a matrix A is written $\det A$ or $|A|$. The determinant is only defined for square matrices.

Fact:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

AND

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Comment: The second formula is called **cofactor expansion**.

Comment: Notice that the second term in the cofactor expansion has a negative sign.

Example: Compute $\det \begin{bmatrix} 1 & 4 & 6 \\ 2 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$

$$= 1 \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 0 & 7 \end{vmatrix} + 6 \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix}$$

$$= 1(-11) - 4(14) + 6(12)$$

$$= 5$$