1. [2 marks] Solve $2x^2y'' - 2xy' + 3y = 0$.

Cauchy-Euler

$$2m(m-1) - 2m + 3 = 0$$

$$2m^2 - 2m - 2m + 3 = 0$$

$$2m^2 - 4m + 3 = 0$$

$$m = \frac{4 \pm \sqrt{-8}}{4}$$

$$m = \frac{4 \pm 2\sqrt{2}\, i}{4}$$

$$m = 1 \pm \frac{\sqrt{2}}{2} i$$

$$y = x\left[C_1 \cos\left(\frac{\sqrt{2}}{2} \ln x\right) + C_2 \sin\left(\frac{\sqrt{2}}{2} \ln x\right)\right]$$

2. [3 marks] a) Set up the DE and the initial conditions.
Do not solve the DE.

A mass weighing 166.6 N stretches a spring by 83.3 cm. The environment creates a damping force with magnitude 4 times velocity. The mass is initially released from 30 cm below the equilibrium position, heading upwards at 2 m/s.

$$mx'' + \beta x' + kx = 0$$

$$\boxed{m = \frac{166.6\,N}{\left(9.8\,\frac{N}{kg}\right)} = 17\,kg}$$

$$\boxed{\beta = 4\,\frac{N}{\left(\frac{m}{s}\right)}}$$

$$\boxed{\begin{aligned} F &= kx \\ 166.6N &= k(0.833m) \\ k &= 200\,\frac{N}{m} \end{aligned}}$$

$$17x'' + 4x' + 200x = 0$$

$\uparrow \ominus$

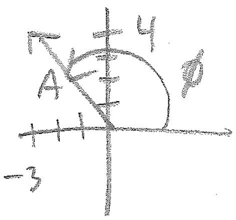
$\downarrow \oplus$

$$x(0) = 0.3\ m$$
$$x'(0) = -2\ \frac{m}{s}$$

2. b) [3 marks] Given $x = 4\cos 2t - 3\sin 2t$, find the first time when the mass passes through the equilibrium position.

$A\cos\phi = -3 \qquad A\sin\phi = 4$

$\vec{A} = [-3, 4]$



$A = \sqrt{9+16}$

$= 5$

$\phi = \tan^{-1}\left(\frac{4}{-3}\right) \quad (+\pi?)$

$= \tan^{-1}\left(-\frac{4}{3}\right) + \pi$

≈ 2.2143

$x = A\sin(2t + \phi)$

$x = 5\sin(2t + 2.2143)$

$x = 0 \Rightarrow \quad 2t + 2.2143 = 0, \pi, 2\pi, \ldots$

$2t + 2.2143 = 0 \Rightarrow t < 0$ nonsense

$2t + 2.2143 = \pi \Rightarrow t \approx 0.46\,s$

3. [6 marks] Solve $y'' - 2y' + y = e^x \ln x$,
given that $y_C = C_1 e^x + C_2 x e^x$.

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= e^x(xe^x + e^x) - xe^{2x}$$

$$= e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ e^x \ln x & \sim \end{vmatrix}$$

$$= -xe^{2x} \ln x$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ \sim & e^x \ln x \end{vmatrix}$$

$$= e^{2x} \ln x$$

$$u_1' = \frac{W_1}{W}$$

$$= -x \ln x$$

$$\begin{array}{c|c} D & I \\ \hline \ln x \; \oplus & -x \\ \frac{1}{x} \; \ominus & -\frac{x^2}{2} \end{array}$$

$$u_1 = \frac{-x^2}{2} \ln x + \int \frac{x}{2}\, dx$$

$$= \frac{-x^2}{2} \ln x + \frac{x^2}{4} \quad \text{(No Constant)}$$

$$u_2' = \frac{W_2}{W}$$
$$= \ln x$$

D		I
$\ln x$	$+$	1
$\frac{1}{x}$	$-$	x

$$u_2 = x\ln x - \int 1\,dx$$
$$= x\ln x - x \quad \text{(No Constant)}$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^x + (x\ln x - x)xe^x$$

or $\left(\frac{1}{2}\ln x - \frac{3}{4}\right)x^2 e^x$

$$y = y_c + y_p$$
$$= C_1 e^x + C_2 x e^x + \left(\frac{1}{2}\ln x - \frac{3}{4}\right)x^2 e^x$$

4. [6 marks] Use sigma notation to write C_2, C_3, C_4 and C_5 in terms of C_0 or C_1:
$y'' - 3xy' = 0$.

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - 3x\sum_{n=1}^{\infty} nC_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=1}^{\infty} 3nC_n x^{n} = 0$$

$$\boxed{\begin{array}{c} k = n-2 \\ n = k+2 \\ n=2 \Rightarrow k=0 \end{array}} \qquad \boxed{\begin{array}{c} k = n \\ n=1 \Rightarrow k=1 \end{array}}$$

Start at largest k-value $(k=1)$.

$$(1^{st}\ \text{term}) + \sum_{k=1}^{\infty} \qquad - \sum_{k=1}^{\infty} \qquad = 0$$

$$2C_2 + \sum_{k=1}^{\infty} (k+2)(k+1)C_{k+2}x^k - \sum_{k=1}^{\infty} 3kC_k x^k = 0$$

$$2C_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} - 3kC_k\right]x^k = 0$$

$$2C_2 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$(k+2)(k+1)C_{k+2} - 3kC_k = 0 \Rightarrow C_{k+2} = \frac{3kC_k}{(k+2)(k+1)}$$

for $k \geq 1$

$\longrightarrow$

$$C_{k+2} = \frac{3kC_k}{(k+2)(k+1)} \quad \text{for } k \geq 1$$

$(k=1)$ $$C_3 = \frac{3C_1}{3 \cdot 2} = \frac{C_1}{2}$$

$(k=2)$ $$C_4 = \frac{6C_2}{4 \cdot 3} = 0$$

$(k=3)$ $$C_5 = \frac{9C_3}{5 \cdot 4} = \frac{9}{20}\left(\frac{C_1}{2}\right) = \frac{9C_1}{40}$$