1. [4 marks] Find an implicit solution:

$\frac{dy}{dx} + (3 + \frac{1}{x})y = \frac{8}{x}$

Linear $\quad P(x) = 3 + \frac{1}{x}$

$$I.F. = e^{\int (3 + \frac{1}{x})\,dx}$$

$$= e^{3x + \ln|x|}$$

$$= e^{3x} \cdot e^{\ln|x|}$$

$$= |x|\, e^{3x}$$

$$= x e^{3x} \qquad (x > 0)$$

$$x e^{3x} \frac{dy}{dx} + (3x + 1) e^{3x} y = 8 e^{3x}$$

Integrate: $\quad x e^{3x} y = \frac{8}{3} e^{3x} + C_1$

or $\quad 3x e^{3x} y = 8 e^{3x} + C$

2. [6 marks] Find an implicit solution:

$[\sin(6y) + xe^x]dx + [6x\cos(6y) + \frac{2}{\sqrt{y}}]dy = 0, \quad y(0) = 4$

$M_y = 6\cos 6y = N_x$

$\Rightarrow$ DE is exact.

$f = \int (\sin 6y + xe^x)\,dx$

$= x\sin 6y + xe^x - e^x + g(y)$

	D	I
+	x	e^x
−	1	e^x
	0	e^x

AND

$f = \int (6x\cos 6y + 2y^{-1/2})\,dy$

$= x\sin 6y + 4y^{1/2} + h(x)$

$\Rightarrow f = x\sin 6y + xe^x - e^x + 4y^{1/2}$

DE: $df = 0$

Solution: $\int df = \int 0$

$f = C$

$x\sin 6y + xe^x - e^x + 4y^{1/2} = C$

Sub $x=0$, $y=4$:

$-1 + 8 = C$

$7 = C$

$\boxed{x\sin 6y + xe^x - e^x + 4\sqrt{y} = 7}$

3. [6 marks] Find an implicit solution:

$\frac{dy}{dx} - \frac{1}{x}y = \frac{x^2}{y}$

Bernoulli $n=-1$

Sub $\begin{cases} y = u^{\frac{1}{1-n}} = u^{1/2} \\ \frac{dy}{dx} = \frac{1}{2}u^{-1/2}\frac{du}{dx} \end{cases}$ (1 MARK)

$\frac{1}{2}u^{-1/2}\frac{du}{dx} - \frac{1}{x}u^{1/2} = x^2 u^{-1/2}$ (1 MARK)

Multiply by $2u^{1/2}$:

$\frac{du}{dx} - \frac{2}{x}u = 2x^2$ Now linear (1 MARK)

$P(x) = -\frac{2}{x}$

$I.F. = e^{\int -\frac{2}{x}\,dx}$

$= e^{-2\ln|x|}$

$= e^{\ln|x|^{-2}}$

$= |x|^{-2}$

$= x^{-2}$ (1 MARK)

$x^{-2}\frac{du}{dx} - 2x^{-3}u = 2$ (1 MARK)

Integrate:

$x^{-2}u = 2x + C$

$x^{-2}y^2 = 2x + C$ or $y^2 = 2x^3 + Cx^2$ (1 MARK)

$y = u^{1/2}$
$y^2 = u$

4. [8 marks] Find an **explicit** solution:

$\frac{dy}{dx} = y^2 - 49$

Separable $\quad \frac{dy}{y^2-49} = dx$

$$\frac{dy}{(y-7)(y+7)} = dx$$

(1) MARK

Partial Fractions

$$\frac{1}{(y-7)(y+7)} = \frac{A}{y-7} + \frac{B}{y+7}$$

$$1 = A(y+7) + B(y-7)$$

Sub $y=-7$: $\quad 1 = -14B \Rightarrow B = -\frac{1}{14}$

Sub $y=7$: $\quad 1 = 14A \Rightarrow A = \frac{1}{14}$

(2) MARKS

$$\frac{1}{14}\left(\frac{1}{y-7} - \frac{1}{y+7}\right) dy = dx$$

$$\int \frac{1}{14}\left(\frac{1}{y-7} - \frac{1}{y+7}\right) dy = \int dx$$

(3) MARKS

$$\frac{1}{14}\left[\ln|y-7| - \ln|y+7|\right] = x + C_1$$

$$\ln|y-7| - \ln|y+7| = 14x + C_2$$

→

$$\ln\left|\frac{y-7}{y+7}\right| = 14x + C_2$$

$e^{LS} = e^{RS}$:

$$\left|\frac{y-7}{y+7}\right| = e^{14x + C_2}$$

$$\frac{y-7}{y+7} = \pm e^{C_2} e^{14x}$$

$$\frac{y-7}{y+7} = Ce^{14x}$$

$$y-7 = Ce^{14x}(y+7)$$

$$y-7 = Ce^{14x}y + 7Ce^{14x}$$

$$y(1-Ce^{14x}) = 7Ce^{14x} + 7$$

$$y = \frac{7(1+Ce^{14x})}{1-Ce^{14x}}$$

(2) MARKS for the explicit solution.