

1. [4 marks] Find all the eigenvalues of $A = \begin{bmatrix} 7 & -2 \\ -1 & 6 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & -2 \\ -1 & 6-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(6-\lambda) - 2 = 0$$

$$42 - 7\lambda - 6\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 13\lambda + 40 = 0$$

$$(\lambda - 5)(\lambda - 8) = 0$$

$$\lambda = 5, 8$$

2. [6 marks] A and B are 3×3 matrices with $\det A = -2$ and $\det B = 7$. State the determinant of the following matrices or write "not enough information".

a) A^{-1}

$$\det A^{-1} = \frac{1}{\det A} = -\frac{1}{2}$$

b) A^T

$$\det A^T = \det A = -2$$

c) $A + B$

Not Enough Information

d) AB

$$\det AB = (\det A)(\det B) = -14$$

e) A^4

$$\det A^4 = (\det A)(\det A)(\det A)(\det A) = 16$$

f) $2B$

$$\det 2B = 2^3 \det B = 56$$

3. [4 marks] $A = \begin{bmatrix} 4 & 8 & 13 & 18 \\ 3 & 6 & 10 & 10 \\ 2 & 4 & 6 & 8 \end{bmatrix}$ has RREF = $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Find a basis for:

a) the row space of A

Nonzero rows of RREF

$$\left\{ \begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

b) the column space of A

Use columns 1, 3 and 4 of A :

$$\left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 13 \\ 10 \\ 6 \end{bmatrix}, \begin{bmatrix} 18 \\ 10 \\ 8 \end{bmatrix} \right\}$$

b) the null space of A

Solve $[A | \vec{0}]$

[RREF of $A | \vec{0}$]

$$\begin{array}{cccc|ccc} x_1 & x_2 & x_3 & x_4 & & & \\ \begin{bmatrix} \textcircled{1} & 2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix} & & & & 0 & 0 & 0 \end{array}$$

↑

$$x_2 = t$$

$$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2t$$

$$x_3 = 0$$

$$x_4 = 0$$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t$$

$$\text{A basis for null}(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

4. [5 marks] Find the standard matrix for:

a) the transformation T where $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -4x + 7y \\ x - 2y \end{bmatrix}$

$$[T] = \begin{bmatrix} -4 & 7 \\ 1 & -2 \end{bmatrix}$$

Coefficients

b) the transformation S where S rotates a vector in \mathbb{R}^2 by 60°

$$[S] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}_{\theta=60^\circ}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

c) the transformation T followed by S

$$[S][T] = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} -4 & 7 \\ 1 & -2 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} -4 - \sqrt{3} & 7 + 2\sqrt{3} \\ -4\sqrt{3} + 1 & 7\sqrt{3} - 2 \end{bmatrix}$$

5. [6 marks] Solve:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -41 \\ 121 \\ -76 \end{bmatrix}$$

$$LU\vec{x} = \vec{b}$$

① Solve $L\vec{y} = \vec{b}$

② Solve $U\vec{x} = \vec{y}$

① $L\vec{y} = \vec{b}$

$$\begin{array}{ccc|c} y_1 & y_2 & y_3 & \\ \hline 1 & 0 & 0 & -41 \\ -3 & 1 & 0 & 121 \\ 2 & 4 & 1 & -76 \end{array}$$

$$y_1 = -41$$

$$-3y_1 + y_2 = 121 \Rightarrow 123 + y_2 = 121 \Rightarrow y_2 = -2$$

$$2y_1 + 4y_2 + y_3 = -76 \Rightarrow -82 - 8 + y_3 = -76 \Rightarrow y_3 = 14$$

② $U\vec{x} = \vec{y}$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 6 & 1 & 3 & -41 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 7 & 14 \end{array}$$

$$7x_3 = 14 \Rightarrow x_3 = 2$$

$$2x_2 - 2x_3 = -2 \Rightarrow 2x_2 - 4 = -2 \Rightarrow x_2 = 1$$

$$6x_1 + x_2 + 3x_3 = -41 \Rightarrow 6x_1 + 1 + 6 = -41 \Rightarrow x_1 = -8$$

$$\vec{x} = \begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix}$$