1. [4 marks] Let $\mathbf{u} = [4, -6]$ and $\mathbf{v} = [9, 3]$. Find:

a) the length of $3\mathbf{u} - 4\mathbf{v}$

$$3\bar{u} - 4\bar{v} = 3\begin{bmatrix} 4 \\ -6 \end{bmatrix} - 4\begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ -18 \end{bmatrix} + \begin{bmatrix} -36 \\ -12 \end{bmatrix}$$

$$= \begin{bmatrix} -24 \\ -30 \end{bmatrix}$$

$$\|3\bar{u} - 4\bar{v}\| = \sqrt{576 + 900}$$

$$= \sqrt{1476}$$

b) the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \, \|\bar{v}\| \cos\theta$$

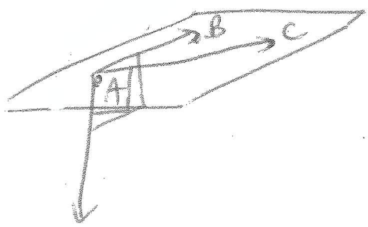
$$18 = \sqrt{52}\,\sqrt{90}\,\cos\theta$$

$$\frac{18}{\sqrt{52}\,\sqrt{90}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{18}{\sqrt{52}\,\sqrt{90}}\right)$$

$$\approx 75^\circ$$

2. [5 marks] Find the general form of the plane that contains the points $A = (6, -2, 1)$, $B = (7, 3, -1)$ and $C = (2, -4, 2)$.



$\vec{n} = \vec{AB} \times \vec{AC}$

$$\vec{AB} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{bmatrix} 1 \\ 7 \\ 18 \end{bmatrix}$$

$$\begin{matrix} 1 & 5 & -2 & 1 & 5 \\ -4 & -2 & 1 & -4 & -2 \end{matrix}$$

Normal form $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$$\begin{bmatrix} 1 \\ 7 \\ 18 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 18 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

General form $\quad x + 7y + 18z = 10$

3. [5 marks] Find the volume of the parallelepiped determined by $\mathbf{u} = [2, -3, 4]$, $\mathbf{v} = [2c, 3, c+1]$, and $\mathbf{w} = [9, 8, 7]$. Simplify your answer. Your answer will involve c.

$$\text{Volume} = \left| \det \begin{bmatrix} 2 & -3 & 4 \\ 2c & 3 & c+1 \\ 9 & 8 & 7 \end{bmatrix} \right|$$

absolute value

$$= \left| 2 \begin{vmatrix} 3 & c+1 \\ 8 & 7 \end{vmatrix} + 3 \begin{vmatrix} 2c & c+1 \\ 9 & 7 \end{vmatrix} + 4 \begin{vmatrix} 2c & 3 \\ 9 & 8 \end{vmatrix} \right|$$

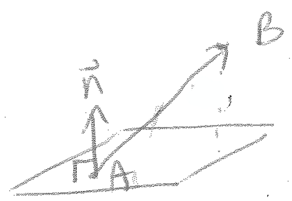
$$= \left| 2[21 - 8(c+1)] + 3[14c - 9(c+1)] + 4[16c - 27] \right|$$

$$= \left| 42 - 16(c+1) + 42c - 27(c+1) + 64c - 108 \right|$$

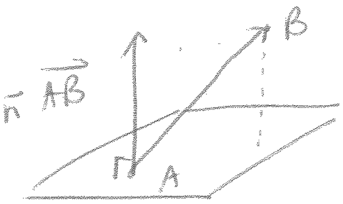
$$= \left| 42 - 16c - 16 + 42c - 27c - 27 + 64c - 108 \right|$$

$$= \left| 63c - 109 \right|$$

4. [5 marks] Find the distance between the plane $4x + 3y + z = 8$ and the point $B = (2, 3, -1)$.



$\text{proj}_{\vec{n}}\,\vec{AB}$



Distance $= \|\text{proj}_{\vec{n}}\,\vec{AB}\|$

A: any point on the plane

$A = (0, 0, 8)$

$\vec{AB} = \begin{bmatrix} 2 \\ 3 \\ -9 \end{bmatrix}$

$\vec{n} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

$$\text{proj}_{\vec{n}}\,\vec{AB} = \frac{\vec{n} \cdot \vec{AB}}{\|\vec{n}\|^2}\,\vec{n}$$

$$= \frac{8}{26} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{4}{13} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

$$\text{Distance} = \left\| \frac{4}{13} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \right\|$$

$$= \frac{4}{13} \left\| \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \right\|$$

$$= \frac{4\sqrt{26}}{13}$$

5. [6 marks] Solve using Gauss-Jordan Elimination:

$$\begin{aligned} x - \ 3y + \ 8z &= \ 7 \\ 2x + \ 6y + 10z &= \ 8 \\ -3x + \ 3y - 21z &= -18 \end{aligned}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 8 & 7 \\ 2 & 6 & 10 & 8 \\ -3 & 3 & -21 & -18 \end{array}\right] \end{array}$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 8 & 7 \\ 0 & 12 & -6 & -6 \\ 0 & -6 & 3 & 3 \end{array}\right]$$

$$\frac{R_2}{12} \left[\begin{array}{ccc|c} 1 & -3 & 8 & 7 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -6 & 3 & 3 \end{array}\right]$$

$$\begin{array}{l} R_1 + 3R_2 \\ \\ R_3 + 6R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{2} & \frac{11}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$z = t$$

$$x + \frac{13}{2} z = \frac{11}{2} \implies x = \frac{11}{2} - \frac{13}{2} t$$

$$y - \frac{1}{2} z = \frac{-1}{2} \implies y = \frac{-1}{2} + \frac{1}{2} t$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/2 \\ -1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -13/2 \\ 1/2 \\ 1 \end{bmatrix}$$