

Math 251 Review Problems

1. Find the angle between $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.
2. Let \mathbf{u} and \mathbf{v} be in \mathbb{R}^n .
Show that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$
3. Find the vector form of the plane $x + y + z = 2$
4. Consider the plane through $(3, 2, 5)$ that is parallel to the plane $2x + 3y - z = 0$. Find the general form of the plane.
5. Find the distance between $Q = (1, 4, 0)$ and the plane $x + 3y + 5z = 0$
6. Find the area of the triangle with vertices $A = (1, 1, 1)$, $B = (2, 4, 6)$ and $C = (3, 5, 10)$.
7. Find a condition on a, b and c so that the system below is consistent:

$$\begin{aligned}x + 5y + z &= a \\2x + y + 3z &= b \\7x - 19y + 13z &= c\end{aligned}$$

8. How many solutions does the following system have?

$$\begin{aligned}x + ky &= 1 \\kx + y &= 1\end{aligned}$$

9. Find the point of intersection of the following lines, if it exists:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

10. Let \mathbf{u}, \mathbf{v} and \mathbf{w} be in \mathbb{R}^n . Show that all three vectors are in $\text{span}(\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w})$

11. Find the general equation of the plane spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

12. Describe the span of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

13. Find the inverse of BB^T for $B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix}$

14. The following sequence of row operations turns the 3×3 matrix A into I . Write A as a product of elementary matrices.

$$R_2 \leftrightarrow R_3, \quad \frac{1}{2}R_3, \quad R_1 \rightarrow R_1 - 3R_3$$

15. Find the LU factorization of A and use it to solve the system below:

$$\begin{aligned} 3x + 5y + \quad &= 23 \\ 6x - 4y + 2z &= -8 \\ 3x - 2y + \quad &= -5 \end{aligned}$$

16. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ has RREF = $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 1 & 5 \\ 4 & 8 & 1 & 6 \end{bmatrix} \text{ has RREF} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for:

- $\text{row}(A)$
- $\text{row}(A)$ consisting of rows of A
- $\text{col}(A)$
- $\text{null}(A)$

17. If A is 4×6 , list the possible values of $\text{nullity}(A)$.

18. Consider the following linear transformations:

S: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a reflection in the x -axis

T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a (counterclockwise) rotation by $\frac{\pi}{3}$

Find $S(T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}))$

19. T is a linear transformation from \mathbb{R}^2 to \mathbb{R}^2 .

Given $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $T(\begin{bmatrix} 6 \\ 8 \end{bmatrix})$.

20. Show that T is not linear:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xy + z \\ x + y + z \end{bmatrix}$$

21. Find y :

$$3x - y + 4z = 32$$

$$5x - 2y - z = 39$$

$$6x + 2y + z = 38$$

22. Do $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ form a basis for \mathbb{R}^3 ?

23. Consider $A = \begin{bmatrix} 4 & -2 & -2 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

a) Find the eigenvalues of A

b) Consider the eigenvalue with the largest algebraic multiplicity. Find a basis for the eigenspace corresponding to this eigenvalue.

24. A is a 2×2 matrix with eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -1$ corresponding to eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find $A^{-5} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

25. $P = \begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix}$ diagonalizes A to produce $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$. Find A^7 .

26. Find the coordinate vector $[\mathbf{v}]_\beta$ of $\mathbf{v} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$ with respect to the orthogonal basis $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ of \mathbb{R}^3 .

27. Find the orthogonal decomposition of $\mathbf{u} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ with respect to $W = \text{span} \left(\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right)$.

28. Find a basis for W^\perp where W is the line $2x - 5y = 0$ in \mathbb{R}^2 .

29. Let $W = \text{span} \left(\begin{bmatrix} 2 \\ 3 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right)$. Find a basis for W^\perp .

30. Find an orthogonal basis for $\text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$.

31. Use the spectral decomposition to find a matrix A with eigenvalue 2 corresponding to eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and eigenvalue -2 corresponding to eigenvector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

32. Find the least squares solution to the following system:

$$\begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

33. Find the least squares regression line for the following set of points: $(1, 2)$, $(2, 3)$, $(3, 5)$ and $(4, 7)$.

34. Express $z = \frac{3-7i}{c+di}$ in rectangular form.

35. Express $z = 5 + 12i$ in polar form.

36. Find all cube roots of -8