1. [4 marks] $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 0 & 1 & 0 & 1 \\ 1 & 3 & 3 & 5 \end{bmatrix}$ has RREF$= \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for:

a) the column space of A

Use Columns 1 and 2 of A.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} \right\}$

b) the row space of A

Use nonzero rows of the RREF.

$\mathcal{B} = \{ [1 \ 0 \ 3 \ 2], [0 \ 1 \ 0 \ 1] \}$

c) the null space of A

Solve $A\vec{x} = \vec{0}$. Each parameter produces a basis vector.

$[A \mid \vec{0}]$

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$x_3 = s$

$x_4 = t$

$x_1 + 3x_3 + 2x_4 = 0 \Rightarrow x_1 = -3s - 2t$

$x_2 + x_4 = 0 \Rightarrow x_2 = -t$

$\vec{x} = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$

$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

2. [5 marks] Let $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -3x + 8y \\ 2x - 7y \end{bmatrix}$.

Let R be the transformation that rotates a vector in $\mathbb{R}^2$ by $30°$ clockwise. Find the standard matrix for the transformation that first performs R and then T. Simplify your answer to a single matrix.

$[T] = \begin{bmatrix} -3 & 8 \\ 2 & -7 \end{bmatrix}$ (the coefficients)

$[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta = -30°}$

$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$

$[T][R] = \frac{1}{2}\begin{bmatrix} -3 & 8 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} -3\sqrt{3} - 8 & -3 + 8\sqrt{3} \\ 2\sqrt{3} + 7 & 2 - 7\sqrt{3} \end{bmatrix}$

3. [5 marks] Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ -8 & 2 & 5 \end{bmatrix}$. Find the LU factorization of A.

$R_1 - \frac{1}{2}R_2$ $\quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 6 & 17 \end{bmatrix} \begin{matrix} \\ k=\frac{1}{2} \\ k=-4 \end{matrix}$

$R_3 + 4R_1$

$$\boxed{\begin{aligned} 1 + ?(2) &= 0 \\ ? &= -\tfrac{1}{2} \end{aligned}}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 19 \end{bmatrix} k=4$$

$R_3 - 4R_2$

$$\boxed{\begin{aligned} 6 + ?\left(\tfrac{3}{2}\right) &= 0 \\ ? &= \frac{-6}{\left(\frac{3}{2}\right)} \\ ? &= -4 \end{aligned}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 19 \end{bmatrix}$$

4. [5 marks] Use Cramer's Rule to find y. Show all your work. (You do not need to find x or z).

$$\begin{aligned} 2x - 3y + 4z &= 53 \\ 2x + y + 2z &= 85 \\ -2x + 2y - 2z &= -34 \end{aligned}$$

$$|A| = \begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 2 \\ -2 & 2 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ -2 & -2 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}$$

$$= 2(-6) + 3(0) + 4(6)$$

$$= 12$$

$$|A_2| = \begin{vmatrix} 2 & 53 & 4 \\ 2 & 85 & 2 \\ -2 & -34 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 85 & 2 \\ -34 & -2 \end{vmatrix} - 53\begin{vmatrix} 2 & 2 \\ -2 & -2 \end{vmatrix} + 4\begin{vmatrix} 2 & 85 \\ -2 & -34 \end{vmatrix}$$

$$= 2(-102) - 53(0) + 4(102)$$

$$= 204$$

$$y = \frac{|A_2|}{|A|}$$

$$= 17$$

5. [6 marks] a) Find all the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-2-\lambda) - 4 = 0$$

$$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\lambda = -3, 2$$

b) Find one eigenvector of $A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$ corresponding to $\lambda = 4$.

Solve $[A - 4I \mid \vec{0}\,]$

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

$\frac{R_1}{4}$ $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 \end{array}\right]$

$R_2 - 2R_1$ $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t$

$x_1 + \frac{1}{2}x_2 = 0 \Rightarrow x_1 = -\frac{1}{2}t$

$$\vec{x} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t \quad (t \neq 0)$$

$\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ or any nonzero multiple of this.