1. [5 marks] Write $C = \begin{bmatrix} 7 & 83 \\ 94 & -179 \end{bmatrix}$ as a linear combination of $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 2 & -5 \end{bmatrix}$, or show that it is not possible to do so.

Let $c_1 A + c_2 B = C$.

$$\begin{array}{cc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 2 & 5 & 83 \\ -3 & 2 & 94 \\ 4 & -5 & -179 \end{array}\right]$$

$R_2 - 2R_1$
$R_3 + 3R_1$
$R_4 - 4R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 3 & 69 \\ 0 & 5 & 115 \\ 0 & -9 & -207 \end{array}\right]$$

$\frac{R_2}{3}$

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 23 \\ 0 & 5 & 115 \\ 0 & -9 & -207 \end{array}\right]$$

$R_1 - R_2$
$R_3 - 5R_2$
$R_4 + 9R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & -16 \\ 0 & 1 & 23 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$c_1 = -16$
$c_2 = 23$

$C = -16A + 23B$

2. [3 marks] Set up a system of equations to balance the following chemical equation. Do not solve the system.

$NH_3 + O_2 \rightarrow NO + H_2O$

$w(NH_3) + x(O_2) \rightarrow y(NO) + z(H_2O)$

N: $w = y$ or $w - y = 0$

H: $3w = 2z$ or $3w - 2z = 0$

O: $2x = y + z$ or $2x - y - z = 0$

or
$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right]$$

3. [7 marks] Let $A = \begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 8 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 7 & 4 \\ -5 & 7 \end{bmatrix}$.

Compute $(A - 2I)B^T + C^2$.

$$A - 2I = \begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 6 & 2 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 8 \\ -2 & 3 \end{bmatrix}$$

$$(A - 2I)B^T = \begin{bmatrix} 1 & -2 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 2 & 8 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 \\ 8 & 54 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 7 & 4 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 7 & 4 \\ -5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 56 \\ -70 & 29 \end{bmatrix}$$

$$(A - 2I)B^T + C^2 = \begin{bmatrix} 6 & 2 \\ 8 & 54 \end{bmatrix} + \begin{bmatrix} 29 & 56 \\ -70 & 29 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 58 \\ -62 & 83 \end{bmatrix}$$

4. [5 marks] Find the general form of span($\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 9 \\ 12 \end{bmatrix}$).

Let $c_1 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

Each zero row of the REF will give one condition.

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 3 & w \\ 0 & 1 & x \\ 3 & 9 & y \\ 2 & 12 & z \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \left[\begin{array}{cc|c} 1 & 3 & w \\ 0 & 1 & x \\ 0 & 0 & y-3w \\ 0 & 6 & z-2w \end{array}\right]$$

$$\begin{array}{l} \\ \\ \\ R_4 - 6R_2 \end{array} \left[\begin{array}{cc|c} 1 & 3 & w \\ 0 & 1 & x \\ 0 & 0 & y-3w \\ 0 & 0 & z-2w-6x \end{array}\right]$$

Consistent system $\Rightarrow$ $y-3w=0$ and $z-2w-6x=0$
$\Rightarrow$ $y=3w$ and $z=2w+6x$

$\left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right.$ such that $y=3w$ and $z=2w+6x \Big\}$

$= \left\{ \begin{bmatrix} w & x \\ 3w & 2w+6x \end{bmatrix} \right\}$

5. [5 marks] Write $A = \begin{bmatrix} 1 & 4 \\ -3 & 6 \end{bmatrix}$ as a product of elementary matrices.

$R_2 + 3R_1$ $\quad \begin{bmatrix} 1 & 4 \\ 0 & 18 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad E_1^{-1} \overset{(R_2 - 3R_1)}{=} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$\frac{R_2}{18}$ $\quad \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{18} \end{bmatrix} \quad E_2^{-1} \overset{(18R_2)}{=} \begin{bmatrix} 1 & 0 \\ 0 & 18 \end{bmatrix}$

$R_1 - 4R_2$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \quad E_3^{-1} \overset{(R_1 + 4R_2)}{=} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$$\underbrace{E_3 E_2 E_1}_{A^{-1}} A = I$$

$$\begin{aligned} A &= (A^{-1})^{-1} \\ &= (E_3 E_2 E_1)^{-1} \\ &= E_1^{-1} E_2^{-1} E_3^{-1} \end{aligned}$$