1. [3 marks] $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Write $\mathbf{w} = \begin{bmatrix} 93 \\ -61 \\ 87 \end{bmatrix}$ as a linear combination of the basis vectors.

Orthogonal basis

$$\Rightarrow \quad \vec{w} = \text{proj}_{\vec{v}_1} \vec{w} + \text{proj}_{\vec{v}_2} \vec{w} + \text{proj}_{\vec{v}_3} \vec{w}$$

$$= \frac{\vec{v}_1 \cdot \vec{w}}{\|\vec{v}_1\|^2} \vec{v}_1 + \ldots$$

$$= \frac{145}{9} \vec{v}_1 - \frac{49}{9} \vec{v}_2 + \frac{395}{9} \vec{v}_3$$

2. [7 marks] Find an orthogonal basis for span($\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$).

Gram-Schmidt Orthogonalization

Partial Basis $X = \left\{ \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \right\}$

$$\vec{v_2} = \begin{bmatrix}1\\1\\2\\3\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\1\\2\\3\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\2\\3\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$$

$$3\vec{v_2} = 3\begin{bmatrix}1\\1\\2\\3\end{bmatrix} - 2\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\3\\8\\7\end{bmatrix}$$ Partial Basis $X = \left\{ \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\8\\7\end{bmatrix} \right\}$

$$\vec{v_3} = \begin{bmatrix}4\\3\\2\\1\end{bmatrix} - \text{proj}_{\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}} \begin{bmatrix}4\\3\\2\\1\end{bmatrix} - \text{proj}_{\begin{bmatrix}1\\3\\8\\7\end{bmatrix}} \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$$

$$= \begin{bmatrix}4\\3\\2\\1\end{bmatrix} - \frac{3}{3}\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} - \frac{36}{123}\begin{bmatrix}1\\3\\8\\7\end{bmatrix}$$

$\frac{12}{41}$

$$41\vec{v_3} = 41\begin{bmatrix}4\\3\\2\\1\end{bmatrix} - 41\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} - 12\begin{bmatrix}1\\3\\8\\7\end{bmatrix}$$

$$= \begin{bmatrix}111\\87\\27\\-84\end{bmatrix}$$

Orthogonal Basis $= \left\{ \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\8\\7\end{bmatrix}, \begin{bmatrix}111\\87\\27\\-84\end{bmatrix} \right\}$

3. [6 marks] Let $W = \text{span}\left(\begin{bmatrix} 1 \\ -3 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 7 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -11 \\ 2 \\ -1 \end{bmatrix} \right)$.

Find a basis for $W^{\perp}$.

Solve $\left[\begin{array}{ccccc|c} 1 & -3 & 5 & 0 & 1 & 0 \\ -1 & 4 & 7 & 1 & 2 & 0 \\ 0 & -1 & -11 & 2 & -1 & 0 \end{array}\right]$

$R_2 + R_1$ $\left[\begin{array}{ccccc|c} 1 & -3 & 5 & 0 & 1 & 0 \\ 0 & 1 & 12 & 1 & 3 & 0 \\ 0 & -1 & -11 & 2 & -1 & 0 \end{array}\right]$

$R_1 + 3R_2$

$R_3 + R_2$ $\left[\begin{array}{ccccc|c} 1 & 0 & 41 & 3 & 10 & 0 \\ 0 & 1 & 12 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \end{array}\right]$

$R_1 - 41R_3$

$R_2 - 12R_3$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -120 & -72 & 0 \\ 0 & 1 & 0 & -35 & -21 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \end{array}\right] \end{array}$$

$x_4 = s \quad x_5 = t$

$x_1 = 120s + 72t$

$x_2 = 35s + 21t$

$x_3 = -3s - 2t$

$$\vec{x} = \begin{bmatrix} 120 \\ 35 \\ -3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 72 \\ 21 \\ -2 \\ 0 \\ 1 \end{bmatrix} t$$

A basis for $W^{\perp}$ is $\left\{ \begin{bmatrix} 120 \\ 35 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 72 \\ 21 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. [7 marks] The matrix A has eigenvalue $\lambda_1 = 2$ corresponding to the eigenvector $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and eigenvalue $\lambda_2 = 3$ corresponding to the eigenvector $\vec{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find $A^3 \begin{bmatrix} 22 \\ 23 \end{bmatrix}$ using the formula $A^n(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1\lambda_1^n\vec{x}_1 + c_2\lambda_2^n\vec{x}_2$.

Let $c_1\bar{x}_1 + c_2\bar{x}_2 = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 3 & 2 & 22 \\ 2 & 3 & 23 \end{array}\right] \end{array}$$

$$\frac{R_1}{3} \quad \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{22}{3} \\ 2 & 3 & 23 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{22}{3} \\ 0 & \frac{5}{3} & \frac{25}{3} \end{array}\right]$$

$$\frac{3}{5} \times R_2 \quad \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{22}{3} \\ 0 & 1 & 5 \end{array}\right]$$

$$R_1 - \frac{2}{3}R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \end{array}\right]$$

$c_1 = 4,\ c_2 = 5$

$$\begin{aligned} A^3\begin{bmatrix} 22 \\ 23 \end{bmatrix} &= A^3\left(4\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) \\ &= c_1\lambda_1^3\bar{x}_1 + c_2\lambda_2^3\bar{x}_2 \\ &= 4(2^3)\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5(3^3)\begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= 32\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 135\begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 366 \\ 469 \end{bmatrix} \end{aligned}$$

5. [7 marks] The matrix A has the eigenvalue 2 corresponding to the eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and the eigenvalue 3 corresponding to the eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find A^n. Simplify your answer as much as possible.

$$P^{-1}AP = D$$

$$\cancel{PP^{-1}}AP = PD$$

$$A\cancel{PP^{-1}} = PDP^{-1}$$

$$A^n = PD\cancel{P^{-1}}\,\cancel{P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1}$$

$$\boxed{A^n = PD^nP^{-1}}$$

$$P = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \qquad P^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$$

$$A^n = PD^nP^{-1}$$

$$= \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 3\cdot 2^n & 2\cdot 3^n \\ 2^{n+1} & 3^{n+1} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 9\cdot 2^n - 4\cdot 3^n & 2\cdot 3^{n+1} - 3\cdot 2^{n+1} \\ 3\cdot 2^{n+1} - 2\cdot 3^{n+1} & 3^{n+2} - 2^{n+2} \end{bmatrix}$$