(1) Let $c_1\vec{u} + c_2\vec{v} = \vec{w}$

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 2 & 4 & 8 \\ 3 & -3 & 21 \\ 5 & 5 & 15 \end{array}\right] \end{array}$$

$$\xrightarrow{\frac{R_1}{2}} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -3 & 21 \\ 5 & 5 & 15 \end{array}\right]$$

$$\xrightarrow{\substack{R_2-3R_1 \\ R_3-5R_1}} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -9 & 9 \\ 0 & -5 & -5 \end{array}\right]$$

$$\xrightarrow{\frac{R_2}{-9}} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & -5 & -5 \end{array}\right]$$

$$\xrightarrow{\substack{R_1-2R_2 \\ R_3+5R_2}} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & -10 \end{array}\right]$$

The system has no solution.

It is not possible to write $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

(2) Let $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$.

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} \\ R_2-2R_1 \\ R_3-5R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ \frac{R_2}{-1} \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1-2R_2 \\ \\ R_3+4R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{RREF}$$

$c_3 = t$

$c_1 - c_3 = 0 \Rightarrow c_1 = c_3 \Rightarrow c_1 = t$

$c_2 + 2c_3 = 0 \Rightarrow c_2 = -2c_3 \Rightarrow c_2 = -2t$

Yes, the vectors are linearly dependent (because the system has infinitely-many solutions).

Choose any nonzero t, say $t=1$:

$c_1 = 1,\ c_2 = -2,\ c_3 = 1$

$\Rightarrow \quad \vec{u} - 2\vec{v} + \vec{w} = \vec{0}$

$\Rightarrow \quad \vec{u} = 2\vec{v} - \vec{w}$ OR $\vec{v} = \frac{1}{2}(\vec{u} + \vec{w})$ OR $\vec{w} = -\vec{u} + 2\vec{v}$

(3) Let x = # of days Mine A must operate

y " B "

z " C "

(tons of anthracite) $9x + 8y + 7z = 334$

(tons of ordinary) $4x + 5y + 4z = 184$

(tons of bituminous) $3x + 6y + 7z = 230$

④

$$A-7I=\begin{bmatrix}6 & -4\\ 1 & 3\end{bmatrix}-7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}6 & -4\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-7 & 0\\ 0 & -7\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -4\\ 1 & -4\end{bmatrix}$$

$$(A-7I)^T=\begin{bmatrix}-1 & 1\\ -4 & -4\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & -5\\ 7 & 4\end{bmatrix}\begin{bmatrix}3 & 5\\ -4 & 1\end{bmatrix}$$

$$=\begin{bmatrix}26 & 5\\ 5 & 39\end{bmatrix}$$

$$(A-7I)^T-BC=\begin{bmatrix}-1 & 1\\ -4 & -4\end{bmatrix}-\begin{bmatrix}26 & 5\\ 5 & 39\end{bmatrix}$$

$$=\begin{bmatrix}-27 & -4\\ -9 & -43\end{bmatrix}$$

(5) Let $c_1 \begin{bmatrix} 1 & 1 \\ 9 & 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ 18 & 6 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & w \\ 1 & 1 & x \\ 9 & 18 & y \\ 4 & 6 & z \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 9R_1 \\ R_4 - 4R_1 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & -1 & x-w \\ 0 & 0 & y-9w \\ 0 & -2 & z-4w \end{array}\right]$$

$$\frac{R_2}{-1} \quad \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & w-x \\ 0 & 0 & y-9w \\ 0 & -2 & z-4w \end{array}\right]$$

$$R_4 + 2R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & w-x \\ 0 & 0 & y-9w \\ 0 & 0 & z-2w-2x \end{array}\right] \text{ REF}$$

$$\boxed{\begin{aligned} & z - 4w + 2(w-x) \\ &= z - 4w + 2w - 2x \\ &= z - 2w - 2x \end{aligned}}$$

Consistent system $\Rightarrow$ $y - 9w = 0$ and $z - 2w - 2x = 0$

$\Rightarrow$ $y = 9w$ and $z = 2w + 2x$

The span is $\left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \text{ such that } y = 9w \text{ and } z = 2w + 2x \right\}$

$= \left\{ \begin{bmatrix} w & x \\ 9w & 2w+2x \end{bmatrix} \right\}$