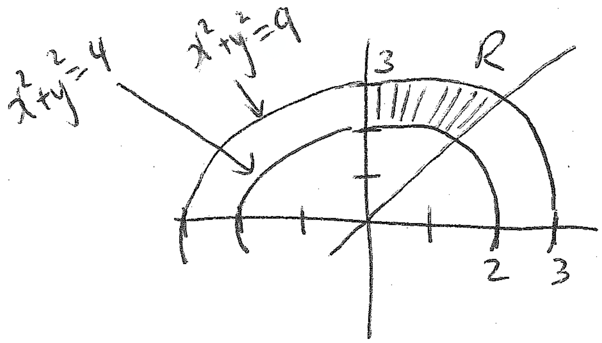
1. [4 marks] Set up a **polar double integral** for the following.
Do not evaluate.

The mass of a thin flat plate with density $= y^2$ where the plate's shape is bounded by $4 \leq x^2 + y^2 \leq 9$, $y \geq 0$ and $0 \leq x \leq y$.



$y = x$

$r\sin\theta = r\cos\theta$

$\tan\theta = 1$

$\theta = \tan^{-1} 1$

$\theta = \frac{\pi}{4}$

$R:$ $\quad 2 \leq r \leq 3$

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$$m = \iint_R \delta \, dA$$

dA = $r\,dr\,d\theta$

δ = $y^2 = (r\sin\theta)^2$

$$m = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_2^3 r^3 \sin^2\theta \, dr\, d\theta$$

2. [6 marks] Ground temperature (in °C) is given by
$f = 0.1x^2y - 0.2xy^2$, where x and y are measured in km.

a) From $A = (4, -2)$, a runner heads towards $B = (7, 8)$. What is the runner's initial rate of change of temperature?

$$\nabla f = [0.2xy - 0.2y^2,\ 0.1x^2 - 0.4xy]$$

$$\nabla f(A) = [-2.4,\ 4.8]$$

$$\text{direction} = \overrightarrow{AB} = [3, 10]$$

$$\bar{u} = \frac{1}{\sqrt{109}}[3, 10]$$

$$D_{\bar{u}} f = \nabla f \cdot \bar{u} = \frac{40.8}{\sqrt{109}} \approx 3.9 \ \frac{°C}{km}$$

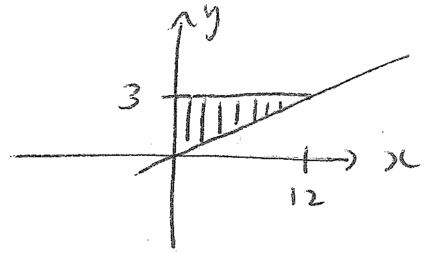
b) Starting from $A = (4, -2)$, in which direction does f increase fastest?

$$\nabla f(A) = [-2.4,\ 4.8]$$

c) Starting from $A = (4, -2)$, what is the maximum rate of increase of f?

$$\|\nabla f(A)\| = \|[-2.4,\ 4.8]\| \approx 5.4 \ \frac{°C}{km}$$

3. [6 marks] Evaluate $\int_0^{12} \int_{x/4}^{3} x\sqrt{1+y^3}\, dy\, dx$.



$y = \frac{x}{4}$

$4y = x$

$x = 4y$

R: $\frac{x}{4} \le y \le 3$

$0 \le x \le 12$

R: $0 \le x \le 4y$

$0 \le y \le 3$

$$\text{Integral} = \int_0^3 \int_0^{4y} x\sqrt{1+y^3}\, dx\, dy$$

$$= \frac{1}{2} \int_0^3 \left[x^2 \sqrt{1+y^3} \right]_{x=0}^{x=4y} dy$$

$$= 8 \int_0^3 y^2 \sqrt{1+y^3}\, dy$$

$$= \frac{8}{3} \int_{y=0}^{y=3} \sqrt{u}\, du$$

Sub $u = 1+y^3$

$du = 3y^2 dy$

$\frac{du}{3} = y^2 dy$

$$= \frac{8}{3} \frac{2}{3} u^{3/2} \Big|_{y=0}^{y=3}$$

$$= \frac{16}{9} (1+y^3)^{3/2} \Big|_0^3$$

$$= \frac{16}{9} \left(28^{3/2} - 1 \right)$$

4. [6 marks] Use the Lagrange Multiplier method to find the point (x, y, z) that minimizes $f = (x-5)^2 + (y+8)^2 + (z-2)^2$, given $2x + 2y - z = 12$.

$$g = 2x + 2y - z$$

$$\nabla f = \lambda \nabla g$$

$$[2(x-5), 2(y+8), 2(z-2)] = \lambda [2, 2, -1]$$

$$2(x-5) = 2\lambda \Rightarrow \lambda = x-5$$

$$2(y+8) = 2\lambda \Rightarrow \lambda = y+8$$

$$2(z-2) = -\lambda \Rightarrow \lambda = -2(z-2)$$

$\lambda = \lambda = \lambda$ (Get y in terms of x and z in terms of x)

$$x-5 = y+8 = -2(z-2)$$

$$x-5 = y+8$$

$$y = x-13$$

$$x-5 = -2(z-2)$$

$$-\tfrac{1}{2}(x-5) = z-2$$

$$z = 2 - \tfrac{1}{2}(x-5)$$

$y, z \rightarrow$ $2x + 2y - z = 12$

$$2x + 2(x-13) - 2 + \tfrac{1}{2}(x-5) = 12$$

$$\tfrac{9}{2}x - \tfrac{61}{2} = 12$$

$$\tfrac{9}{2}x = \tfrac{85}{2}$$

$$x = \tfrac{85}{9}$$

$\Rightarrow y = x - 13 = \frac{-32}{9}$ and $z = 2 - \frac{1}{2}(x-5) = \frac{-2}{9}$