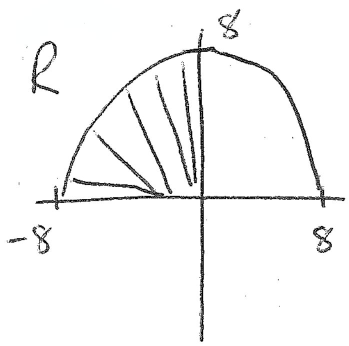
1. [4 marks] Set up a **polar double integral** for the following.
Do not evaluate.

The volume under $z = x^2y^2$ over the region bounded by $0 \le y \le \sqrt{64 - x^2}$ and $-8 \le x \le 0$.



$y = \sqrt{64 - x^2}$

$y^2 = 64 - x^2$

$x^2 + y^2 = 64$

$y = \sqrt{64 - x^2}$ is the upper semicircle.

R: $0 \le r \le 8$

$\frac{\pi}{2} \le \theta \le \pi$

$$V = \iint_R z \, dA$$

$r \, dr \, d\theta$

x^2y^2

$= (r\cos\theta)^2 (r\sin\theta)^2$

$$V = \int_{\frac{\pi}{2}}^{\pi} \int_0^8 r^5 \cos^2\theta \sin^2\theta \, dr \, d\theta$$

2. [6 marks] Ground temperature (in °C) is given by $f = 0.2x^2y - 0.1xy^2$, where x and y are measured in km.

a) From $A = (3, -3)$, a runner heads towards $B = (6, 5)$. What is the runner's initial rate of change of temperature?

$$\nabla f = [0.4xy - 0.1y^2,\ 0.2x^2 - 0.2xy]$$

$$\nabla f(A) = [-4.5,\ 3.6]$$

$$\text{direction} = \vec{AB} = [3, 8]$$

$$\vec{u} = \frac{1}{\sqrt{73}}[3, 8]$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{15.3}{\sqrt{73}} \approx 1.8 \ \frac{°C}{km}$$

b) Starting from $A = (3, -3)$, in which direction does f increase fastest?

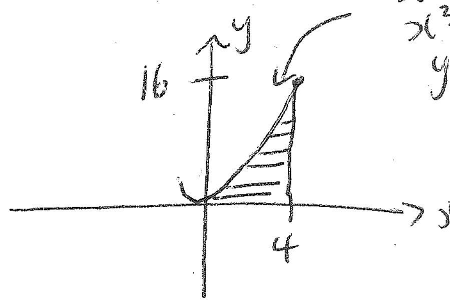
$$\nabla f(A) = [-4.5, 3.6]$$

c) Starting from $A = (3, -3)$, what is the maximum rate of increase of f?

$$\|\nabla f(A)\| = \|[-4.5, 3.6]\| \approx 5.8 \ \frac{°C}{km}$$

3. [6 marks] Evaluate $\int_0^{16} \int_{\sqrt{y}}^{4} \sqrt{1+x^3}\, dx\, dy$.

$x = \sqrt{y}$
$x^2 = y$
$y = x^2$



$$R: \quad \sqrt{y} \le x \le 4$$
$$0 \le y \le 16$$

$$R: \quad 0 \le y \le x^2$$
$$0 \le x \le 4$$

$$\text{Integral} = \int_0^4 \int_0^{x^2} \sqrt{1+x^3}\; dy\, dx$$

$$= \int_0^4 \Big[y\sqrt{1+x^3} \Big]_{y=0}^{y=x^2} dx$$

$$= \int_0^4 x^2 \sqrt{1+x^3}\; dx$$

Sub $u = 1 + x^3$
$du = 3x^2\,dx$
$\frac{du}{3} = x^2\,dx$

$$= \frac{1}{3} \int_{x=0}^{x=4} \sqrt{u}\; du$$

$$= \frac{1}{3}\,\frac{2}{3}\, u^{3/2} \Big|_{x=0}^{x=4}$$

$$= \frac{2}{9} \sqrt{1+x^3}^{\,3} \Big|_{x=0}^{x=4}$$

$$= \frac{2}{9}\left(65^{3/2} - 1\right)$$

4. [6 marks] Use the Lagrange Multiplier method to find the point (x, y, z) that minimizes $f = (x-3)^2 + (y+6)^2 + (z-7)^2$, given $2x + 2y - z = 12$.

$$g = 2x + 2y - z$$

$$\nabla f = \lambda \nabla g$$

$$[2(x-3), 2(y+6), 2(z-7)] = \lambda [2, 2, -1]$$

$$2(x-3) = 2\lambda \Rightarrow \lambda = x-3$$

$$2(y+6) = 2\lambda \Rightarrow \lambda = y+6$$

$$2(z-7) = -\lambda \Rightarrow \lambda = -2(z-7)$$

$\lambda = \lambda = \lambda$ (Get y and z both in terms of x)

$$x-3 = y+6 = -2(z-7)$$

$$x-3 = y+6$$

$$y = x-9$$

$$x-3 = -2(z-7)$$

$$-\tfrac{1}{2}(x-3) = z-7$$

$$z = 7 - \tfrac{1}{2}(x-3)$$

$y, z \rightarrow$

$$2x + 2y - z = 12$$

$$2x + 2(x-9) - [7 - \tfrac{1}{2}(x-3)] = 12$$

$$\frac{9}{2}x - \frac{53}{2} = 12$$

$$\frac{9}{2}x = \frac{77}{2}$$

$$x = \frac{77}{9}$$

$$\Rightarrow y = x - 9 = -\frac{4}{9} \quad \text{and} \quad z = 7 - \tfrac{1}{2}(x-3) = \frac{38}{9}$$