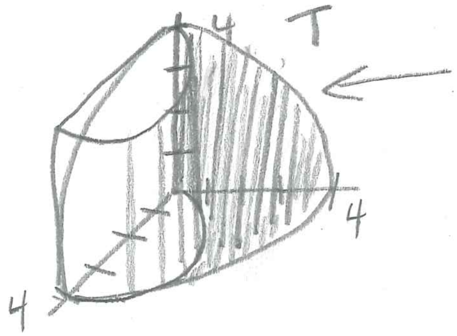
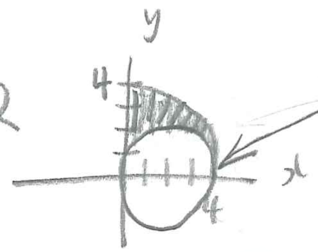
1. [4 marks] Use a triple integral in cylindrical coordinates to set up the volume of the solid region in the first octant inside $z = \sqrt{16 - x^2 - y^2}$ and outside $x^2 + y^2 = 4x$. Do not evaluate the integral.



$$z = \sqrt{16 - x^2 - y^2}$$
$$z = \sqrt{16 - r^2}$$

R



$$x^2 + y^2 = 4x$$
$$r^2 = 4r\cos\theta$$
$$r = 4\cos\theta$$

T:
$$0 \le z \le \sqrt{16 - r^2}$$
$$4\cos\theta \le r \le 4$$
$$0 \le \theta \le \frac{\pi}{2}$$

$$V = \iiint_T dV$$
$$= \int_0^{\frac{\pi}{2}} \int_{4\cos\theta}^{4} \int_0^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta$$

2. [5 marks] Find the work done by the force field $\mathbf{F} = [3y^2,\ 2xy]$ in moving a particle over the line segment from $(-1, 0)$ to $(5, 7)$.

$$W = \int_C (P\,dx + Q\,dy)$$

$$= \int_C [\,3y^2\,dx + 2xy\,dy\,]$$

$$\boxed{\begin{array}{lll} C: & x = -1+6t & dx = 6\,dt \\ & y = 7t & dy = 7\,dt \\ & 0 \le t \le 1 & \end{array}}$$

$$= \int_0^1 [\,3(49t^2)(6dt) + \underbrace{2(-1+6t)(7t)(7dt)}_{98t(-1+6t)dt}\,]$$

$$= \int_0^1 [\,\underbrace{882t^2 - 98t + 588t^2}_{1470t^2 - 98t}\,]dt$$

$$= [490t^3 - 49t^2]_0^1$$

$$= 441 \text{ J}$$

3. [5 marks] Use a change of variables to evaluate $I = \iint\limits_R \frac{(2x+y)^2}{6+3x-y} dA$ where R is bounded by $2x+y=2,\ 2x+y=4,\ 3x-y=0$ and $3x-y=1$.

Let $u = 2x+y$, $\quad 2 \le u \le 4$

$v = 3x-y$, $\quad 0 \le v \le 1$

$$\text{Integrand} = \frac{(2x+y)^2}{6+3x-y} = \frac{u^2}{6+v}$$

$$J_{(u,v)\to(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -5$$

$$J_{(x,y)\to(u,v)} = \frac{-1}{5}$$

$$|J_{(x,y)\to(u,v)}| = \frac{1}{5}$$

$$I = \int_0^1 \int_2^4 \frac{u^2}{6+v} \underbrace{\frac{1}{5}}_{|J_{(x,y)\to(u,v)}|} \, du\, dv$$

$$= \frac{1}{5}\left[\frac{u^3}{3}\right]_2^4 \int_0^1 \frac{1}{6+v}\, dv$$

$$= \frac{1}{5}\left[\frac{64}{3} - \frac{8}{3}\right]\left[\ln|6+v|\right]_0^1$$

$$= \frac{56}{15}\left[\ln 7 - \ln 6\right]$$

4. [6 marks] Find the surface area of the following parametric surface:
$\mathbf{r} = [2\sin a \cos b,\ 2\sin a \sin b,\ 2\cos a]$ where $0 \le a \le \pi$, $0 \le b \le 2\pi$.

$$\vec{r}_a = [2\cos a\cos b,\ 2\cos a\sin b,\ -2\sin a]$$

$$\vec{r}_b = [-2\sin a\sin b,\ 2\sin a\cos b,\ 0]$$

$$\vec{r}_a \times \vec{r}_b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos a\cos b & 2\cos a\sin b & -2\sin a \\ -2\sin a\sin b & 2\sin a\cos b & 0 \end{vmatrix}$$

$$= \vec{i}\,[4\sin^2 a\cos b]$$
$$-\vec{j}\,[-4\sin^2 a\sin b]$$
$$+\vec{k}\,[4\sin a\cos a\cos^2 b + 4\sin a\cos a\sin^2 b]$$

$$= [4\sin^2 a\cos b,\ 4\sin^2 a\sin b,\ 4\sin a\cos a\underbrace{(\cancel{\cos^2 b+\sin^2 b})}_{1}]$$

$$\|\vec{r}_a\times\vec{r}_b\| = \sqrt{16\sin^4 a\cos^2 b + 16\sin^4 a\sin^2 b + 16\sin^2 a\cos^2 a}$$

$$= \sqrt{16\sin^4 a\,(\underbrace{\cancel{\cos^2 b+\sin^2 b}}_{1}) + 16\sin^2 a\cos^2 a}$$

$$= \sqrt{16\sin^2 a\,(\underbrace{\cancel{\sin^2 a+\cos^2 a}}_{1})}$$

$$= |4\sin a| \qquad \text{but } 0 \le a \le \pi$$

$$= 4\sin a$$

$\rightarrow$

$$SA = \iint_R \|\vec{r}_a \times \vec{r}_b\| \, da \, db$$

$$= \int_0^{2\pi}\int_0^{\pi} 4\sin a \, da \, db$$

$$= \left[-4\cos a\right]_0^{\pi} \int_0^{2\pi} db$$

$$= \left[4-(-4)\right]\left[b\right]_0^{2\pi}$$

$$= 8(2\pi)$$

$$= 16\pi$$