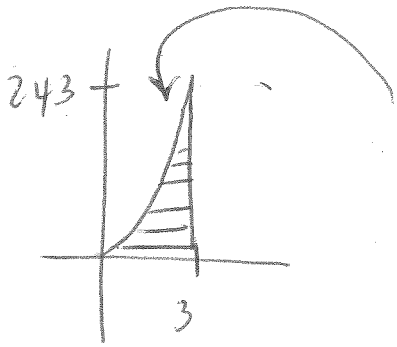
1. [3 marks] Rewrite the integral using vertical slices instead of horizontal slices. **Do not evaluate the integral.**

$$\int_0^{243} \int_{\sqrt[5]{y}}^{3} \sqrt{1+x^6}\, dx\, dy$$

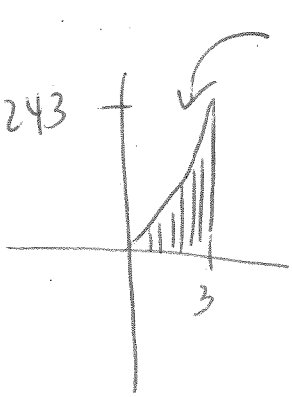
$R: \quad \sqrt[5]{y} \le x \le 3$

$0 \le y \le 243$




$\sqrt[5]{y} = x$

$y = x^5$



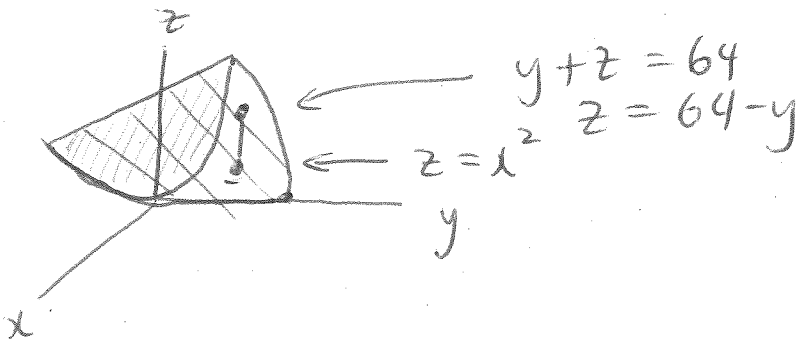

$y = x^5$

$R: \quad 0 \le y \le x^5$

$0 \le x \le 3$

$$\text{Integral} = \int_0^3 \int_0^{x^5} \sqrt{1+x^6}\, dy\, dx$$

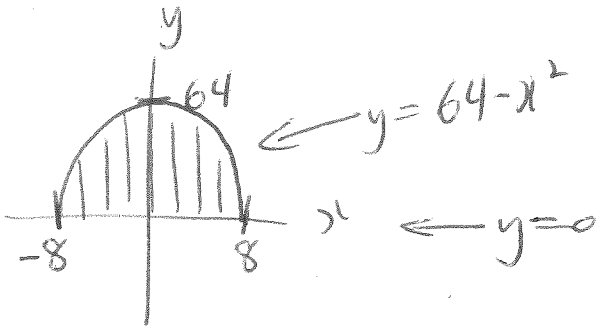
2. [4 marks] Set up a triple integral for the volume of the solid bounded by $z = x^2$, $y + z = 64$, and $y = 0$. **Do not evaluate the integral.**




Slice in z-direction:

$$x^2 \le z \le 64 - y$$

Project on xy-plane:




Eliminate z

$64 - y = x^2$

$64 - x^2 = y$

$y = 64 - x^2$

$\begin{cases} y = 64 - x^2 \\ y = 0 \end{cases}$

$R: \quad 0 \le y \le 64 - x^2$

$-8 \le x \le 8$

$$V = \int_{-8}^{8} \int_{0}^{64-x^2} \int_{x^2}^{64-y} dz\, dy\, dx$$

3. [6 marks] $T = 0.03x^2y - 0.04xy^2$ gives temperature (in °C) in a small flat town. The variables $x$ and $y$ represent position (in km).

a) A runner travels in a straight line from $(x, y) = (2, -4)$ to $(x, y) = (8, 7)$. What initial rate of change of temperature does the runner experience?

$\nabla T = [0.06xy - 0.04y^2,\ 0.03x^2 - 0.08xy]$

$\nabla T(2,-4) = [-1.12,\ 0.76]$

direction $= [6, 11]$

$\vec{u} = \frac{1}{\sqrt{157}}[6, 11]$

$D_{\vec{u}} T = \frac{1}{\sqrt{157}}[6, 11] \cdot [-1.12, 0.76]$

$= \frac{1.64}{\sqrt{157}}$

$\simeq 0.13 \ \frac{°C}{km}$

b) Starting from $(x, y) = (2, -4)$, in which direction does the temperature increase fastest?

$[-1.12, 0.76]$ or any positive multiple of this

c) Starting from $(x, y) = (2, -4)$, what is the maximum rate of change of temperature the runner could experience?

$\|[-1.12, 0.76]\| \simeq 1.35 \ \frac{°C}{km}$

4. [6 marks] Solve the following problem using the method of **Lagrange Multipliers**. Given $2x - 2y + 2z = 7$, find the values of $x, y$, and $z$ that minimize $f = (x-2)^2 + (y+1)^2 + (z+2)^2$.

Let $g = 2x - 2y + 2z$

$$\nabla f = \lambda \nabla g$$

$$[2(x-2),\ 2(y+1),\ 2(z+2)] = \lambda [2, -2, 2]$$

$$2(x-2) = 2\lambda \Rightarrow \lambda = x - 2$$

$$2(y+1) = -2\lambda \Rightarrow \lambda = -(y+1)$$

$$2(z+2) = 2\lambda \Rightarrow \lambda = z + 2$$

$$\lambda = \lambda = \lambda$$

$$x - 2 = -(y+1) = z + 2$$

$$x - 2 = -y - 1$$

$$y = 1 - x$$

$$x - 2 = z + 2$$

$$x - y = z$$

$$z = x - y$$

$$\left.\begin{array}{l} y = 1 - x \\ z = x - y \end{array}\right\} \rightarrow 2x - 2y + 2z = 7$$

$$2x - 2(1-x) + 2(x-y) = 7$$

$$6x - 10 = 7$$

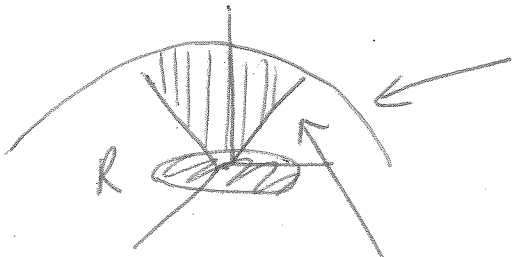
$$6x = 17$$

$$\boxed{x = \frac{17}{6}}$$

$$\boxed{y = 1 - x = \frac{6}{6} - \frac{17}{6} = \frac{-11}{6}}$$

$$\boxed{z = x - y = \frac{17}{6} - \frac{24}{6} = \frac{-7}{6}}$$

5. [6 marks] Use a **double integral** in **polar coordinates** to calculate the volume between the cone $z = \sqrt{x^2 + y^2}$ and the hemisphere $z = \sqrt{2 - x^2 - y^2}$.




$z = \sqrt{2 - x^2 - y^2}$

$z = \sqrt{2 - r^2}$

$z = \sqrt{x^2 + y^2}$

$z = r$

R: $z = z$

$\sqrt{2 - r^2} = r$

$2 - r^2 = r^2$

$2 = 2r^2$

$r^2 = 1$

~~$r = \pm 1$~~ $r = 1$

R: $0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

$$V = \iint_R (z_{top} - z_{bottom})\, dA$$

$$= \int_0^{2\pi} \int_0^1 \left(\sqrt{2 - r^2} - r\right) r\, dr\, d\theta \rightarrow$$

$$= \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} - r^2\right) dr\, d\theta$$

Let $I = \int r\sqrt{2-r^2}\, dr$

$u = 2 - r^2$

$du = -2r\, dr$

$-\frac{1}{2} du = r\, dr$

$I = -\frac{1}{2} \int \sqrt{u}\, du$

$= -\frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$

$= -\frac{1}{3}(2-r^2)^{3/2} + C$

$$= \int_0^{2\pi} \left[ -\frac{1}{3}(2-r^2)^{3/2} - \frac{r^3}{3} \right]_0^1 d\theta$$

$$= \left[ -\frac{1}{3} - \frac{1}{3} + \frac{1}{3}(2)^{3/2} \right] \int_0^{2\pi} d\theta$$

$$= \frac{2\sqrt{2} - 2}{3} \underbrace{\theta \Big|_0^{2\pi}}_{2\pi}$$

$$= \frac{4(\sqrt{2} - 1)\pi}{3}$$