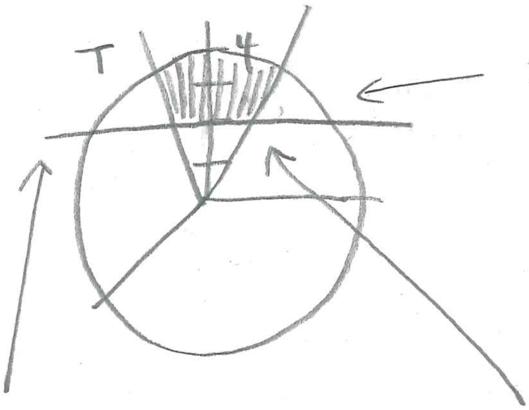
1. [4 marks] Use a triple integral in spherical coordinates to set up the volume of the solid region inside $z = 2\sqrt{x^2+y^2}$, inside $x^2+y^2+z^2 = 16$, and above $z = 2$. Do not evaluate the integral.



$x^2+y^2+z^2 = 16$

$\rho^2 = 16$

$\cancel{\rho = \pm 4}$ $\rho = 4$

$z = 2$

$\rho\cos\phi = 2$

$\rho = 2\sec\phi$

$z = 2\sqrt{x^2+y^2}$

$z = 2r$

$\rho\cos\phi = 2\rho\sin\phi$

$\frac{1}{2} = \tan\phi$

$\phi = \arctan\frac{1}{2}$

T: $2\sec\phi \le \rho \le 4$

$0 \le \phi \le \arctan\frac{1}{2}$

$0 \le \theta \le 2\pi$

$$V = \iiint_T dV$$

$$= \int_0^{2\pi}\int_0^{\arctan\frac{1}{2}}\int_{2\sec\phi}^{4} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

2. [5 marks] Find the work done by the force field $\mathbf{F} = [2xy,\ 3y^2]$ in moving a particle over the line segment from $(-1, 0)$ to $(5, 7)$.

$$W = \int_C (P\,dx + Q\,dy)$$

$$\boxed{\begin{array}{lll} C: & x = -1 + 6t & dx = 6\,dt \\ & y = 7t & dy = 7\,dt \\ & 0 \le t \le 1 & \end{array}}$$

$$= \int_C [2xy\,dx + 3y^2\,dy]$$

$$= \int_0^1 [\underbrace{2(-1+6t)(7t)(6dt)}_{84t(-1+6t)dt} + 3(49t^2)7dt]$$

$$= \int_0^1 [-84t + \underbrace{504t^2 + 1029t^2}_{1533t^2}]\,dt$$

$$= [-42t^2 + 511t^3]_0^1$$

$$= 469 \text{ J}$$

3. [5 marks] Use a change of variables to evaluate $I = \iint\limits_R \frac{(2x+y)^3}{2+3x-y} dA$ where R is bounded by $2x + y = 2,\ 2x + y = 3,\ 3x - y = 0$ and $3x - y = 1$.

Let $u = 2x+y$, $\quad 2 \leq u \leq 3$

$v = 3x - y \quad 0 \leq v \leq 1$

$$\text{Integrand} = \frac{(2x+y)^3}{2+3x-y} = \frac{u^3}{2+v}$$

$$J_{(u,v)\to(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= -5$$

$$J_{(x,y)\to(u,v)} = \frac{-1}{5}$$

$$|J_{(x,y)\to(u,v)}| = \frac{1}{5}$$

$$I = \int_0^1 \int_2^3 \frac{u^3}{2+v} \underbrace{\frac{1}{5}}_{|J_{(x,y)\to(u,v)}|} du\, dv$$

$$= \frac{1}{5}\left[\frac{u^4}{4}\right]_2^3 \int_0^1 \frac{1}{2+v} dv$$

$$= \frac{1}{5}\left(\frac{81}{4} - \frac{16}{4}\right)\left[\ln|2+v|\right]_0^1$$

$$= \frac{13}{4} \left[\ln 3 - \ln 2\right]$$

$$\approx 1.3178$$

4. [6 marks] Find the surface area of the following parametric surface:
$\mathbf{r} = [3\sin a \cos b,\ 3\sin a \sin b,\ 3\cos a]$ where $0 \le a \le \pi$, $0 \le b \le 2\pi$.

$$\vec{r}_a = [3\cos a\cos b,\ 3\cos a\sin b,\ -3\sin a]$$

$$\vec{r}_b = [-3\sin a\sin b,\ 3\sin a\cos b,\ 0]$$

$$\vec{r}_a \times \vec{r}_b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos a\cos b & 3\cos a\sin b & -3\sin a \\ -3\sin a\sin b & 3\sin a\cos b & 0 \end{vmatrix}$$

$$= \vec{i}\,[9\sin^2 a\cos b]$$
$$-\vec{j}\,[-9\sin^2 a\sin b]$$
$$+\vec{k}\,[9\sin a\cos a\cos^2 b + 9\sin a\cos a\sin^2 b]$$

$$= [9\sin^2 a\cos b,\ 9\sin^2 a\sin b,\ 9\sin a\cos a\,\underbrace{\cancel{(\cos^2 b+\sin^2 b)}}_{1}]$$

$$\|\vec{r}_a \times \vec{r}_b\| = \sqrt{81\sin^4 a\cos^2 b + 81\sin^4 a\sin^2 b + 81\sin^2 a\cos^2 a}$$

$$= \sqrt{81\sin^4 a\,\underbrace{\cancel{(\cos^2 b+\sin^2 b)}}_{1} + 81\sin^2 a\cos^2 a}$$

$$= \sqrt{81\sin^2 a\,\underbrace{\cancel{(\sin^2 a+\cos^2 a)}}_{1}}$$

$$= |9\sin a| \quad \text{but} \quad 0 \le a \le \pi$$

$$= 9\sin a$$

$\longrightarrow$

$$SA = \iint_R \|\vec{r}_a \times \vec{r}_b\| \, da\, db$$

$$= \int_0^{2\pi}\int_0^{\pi} 9\sin a \, da\, db$$

$$= [-9\cos a]_0^{\pi} \int_0^{2\pi} db$$

$$= [9+9]\,[b]_0^{2\pi}$$

$$= 18(2\pi)$$

$$= 36\pi$$