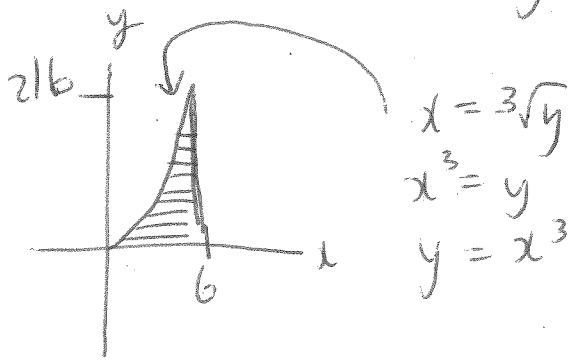
1. [3 marks] Rewrite the integral using vertical slices instead of horizontal slices. **Do not evaluate the integral.**

$$\int_0^{216} \int_{\sqrt[3]{y}}^{6} \sqrt{1+x^4}\, dx\, dy$$

R: $\sqrt[3]{y} \le x \le 6$

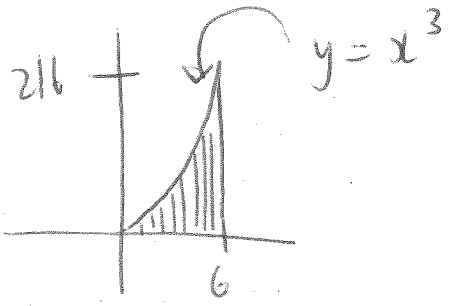
$0 \le y \le 216$



$x = \sqrt[3]{y}$

$x^3 = y$

$y = x^3$

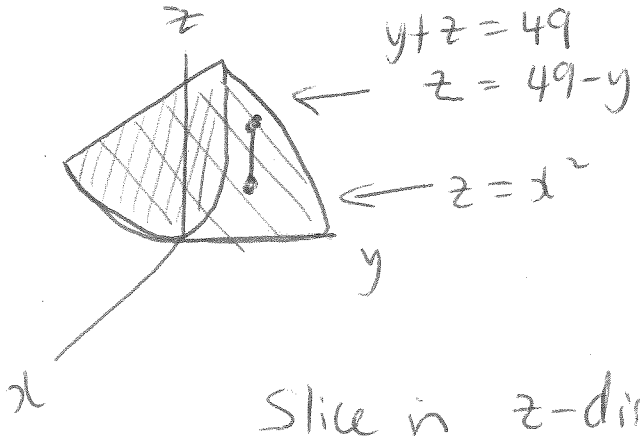


R: $0 \le y \le x^3$

$0 \le x \le 6$

$$\text{Integral} = \int_0^{6} \int_0^{x^3} \sqrt{1+x^4}\, dy\, dx$$

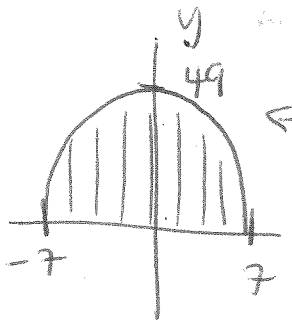
2. [4 marks] Set up a triple integral for the volume of the solid bounded by $z = x^2$, $y + z = 49$, and $y = 0$. **Do not evaluate the integral.**



Slice in z-direction:

$$x^2 \le z \le 49 - y$$

Project on xy-plane:



Eliminate z

$$z = z$$
$$x^2 = 49 - y$$
$$y = 49 - x^2$$

$$\begin{cases} y = 49 - x^2 \\ y = 0 \end{cases}$$

$$R: \quad 0 \le y \le 49 - x^2$$
$$-7 \le x \le 7$$

$$V = \int_{-7}^{7} \int_{0}^{49-x^2} \int_{x^2}^{49-y} dz\, dy\, dx$$

3. [6 marks] $T = 0.03x^2y - 0.04xy^2$ gives temperature (in °C) in a small flat town. The variables x and y represent position (in km).

a) A runner travels in a straight line from $(x, y) = (-3, 1)$ to $(x, y) = (8, 7)$. What initial rate of change of temperature does the runner experience?

$\nabla T = [0.06xy - 0.04y^2,\ 0.03x^2 - 0.08xy]$

$\nabla T(-3,1) = [-0.22,\ 0.51]$

direction $= [11, 6]$

$\vec{u} = \frac{1}{\sqrt{157}}[11, 6]$

$D_{\vec{u}} T = \frac{1}{\sqrt{157}}[11,6] \cdot [-0.22, 0.51]$

$= \frac{0.64}{\sqrt{157}}$

$\approx 0.05 \frac{°C}{km}$

b) Starting from $(x, y) = (-3, 1)$, in which direction does the temperature increase fastest?

$[-0.22, 0.51]$ or any positive multiple of this

c) Starting from $(x, y) = (-3, 1)$, what is the maximum rate of change of temperature the runner could experience?

$\|[-0.22, 0.51]\| \approx 0.56 \frac{°C}{km}$

4. [6 marks] Solve the following problem using the method of **Lagrange Multipliers**. Given $2x - 2y + 2z = 9$, find the values of x, y, and z that minimize $f = (x-2)^2 + (y+1)^2 + (z+2)^2$.

Let $g = 2x - 2y + 2z$

$\nabla f = \lambda \nabla g$

$[2(x-2), 2(y+1), 2(z+2)] = \lambda [2, -2, 2]$

$2(x-2) = 2\lambda \Rightarrow \lambda = x - 2$

$2(y+1) = -2\lambda \Rightarrow \lambda = -(y+1)$

$2(z+2) = 2\lambda \Rightarrow \lambda = z + 2$

$\lambda = \lambda = \lambda$

$x - 2 = -(y+1) = z + 2$

$x - 2 = -y - 1$

$y = 1 - x$

$x - 2 = z + 2$

$x - 4 = z$

$z = x - 4$

$y = 1 - x$
$z = x - 4$ $\Big\} \rightarrow$ $2x - 2y + 2z = 9$

$2x - 2(1-x) + 2(x-4) = 9$

$6x - 10 = 9$

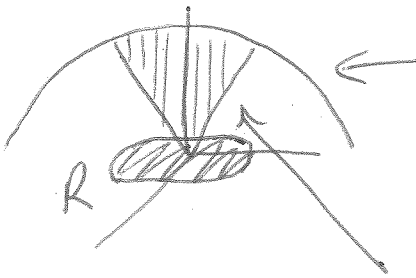
$6x = 19$

$\boxed{x = \frac{19}{6}}$

$\boxed{y = 1 - x = \frac{6}{6} - \frac{19}{6} = \frac{-13}{6}}$

$\boxed{z = x - 4 = \frac{19}{6} - \frac{24}{6} = \frac{-5}{6}}$

5. [6 marks] Use a **double integral** in **polar coordinates** to calculate the volume between the cone $z = \sqrt{x^2 + y^2}$ and the hemisphere $z = \sqrt{2 - x^2 - y^2}$.



$z = \sqrt{2 - x^2 - y^2}$

$z = \sqrt{2 - r^2}$

$z = \sqrt{x^2 + y^2}$

$z = r$

R: $z = z$

$\sqrt{2 - r^2} = r$

$2 - r^2 = r^2$

$2 = 2r^2$

$r^2 = 1$

~~$r = \pm 1$~~ $r = 1$

R: $0 \le r \le 1$
$0 \le \theta \le 2\pi$

$$V = \iint_R (z_{top} - z_{bottom})\, dA$$

$$= \int_0^{2\pi} \int_0^1 \left(\sqrt{2 - r^2} - r\right) r\, dr\, d\theta$$

→

$$= \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} - r^2\right) dr\, d\theta$$

Let $I = \int r\sqrt{2-r^2}\, dr$

$u = 2 - r^2$

$du = -2r\, dr$

$-\frac{1}{2} du = r\, dr$

$I = -\frac{1}{2}\int \sqrt{u}\, du$

$= -\frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$

$= -\frac{1}{3}(2-r^2)^{3/2} + C$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^3}{3} \right]_0^1 d\theta$$

$$= \left[-\frac{1}{3} - \frac{1}{3} + \frac{1}{3}(2)^{3/2} \right] \int_0^{2\pi} d\theta$$

$$= \frac{2\sqrt{2} - 2}{3} \underbrace{\theta \Big|_0^{2\pi}}_{2\pi}$$

$$= \frac{4(\sqrt{2} - 1)\pi}{3}$$