1. [6 marks] $T = 0.02x^2y - 0.03xy^2$ gives temperature (in °C) in a small flat town. The variables x and y represent position (in km).

a) A runner travels from $(x, y) = (5, -4)$ to $(x, y) = (7, 1)$. What initial rate of change of temperature does the runner experience?

direction $= [2, 5]$

$$\vec{u} = \frac{1}{\sqrt{29}}[2,5]$$

$$\nabla T = [0.04xy - 0.03y^2, \; 0.02x^2 - 0.06xy]$$

$$\nabla T(5,-4) = [-1.28, 1.7]$$

$$D_{\vec{u}} T = \nabla T \cdot \vec{u}$$

$$= \frac{1}{\sqrt{29}}[-1.28, 1.7] \cdot [2, 5]$$

$$= \frac{5.94}{\sqrt{29}}$$

$$\approx 1.10 \; \frac{°C}{km}$$

b) Starting from $(x, y) = (5, -4)$, what is the maximum rate of change of temperature the runner could experience?

$$\|[-1.28, 1.7]\| \approx 2.13 \; \frac{°C}{km}$$

c) Starting from $(x, y) = (5, -4)$, in which direction does the temperature increase fastest?

$[-1.28, 1.7]$ or any positive multiple of this

2. [4 marks] Evaluate: $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \int_{3}^{4+\sin\theta} r \, dr \, d\theta$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} r^2 \Big|_{r=3}^{r=4+\sin\theta} d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \left[(4+\sin\theta)^2 - 9\right] d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (16 + 8\sin\theta + \underbrace{\sin^2\theta}_{\frac{1}{2} - \frac{\cos 2\theta}{2}} - 9) \, d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \left(\frac{15}{2} + 8\sin\theta - \frac{\cos 2\theta}{2}\right) d\theta$$

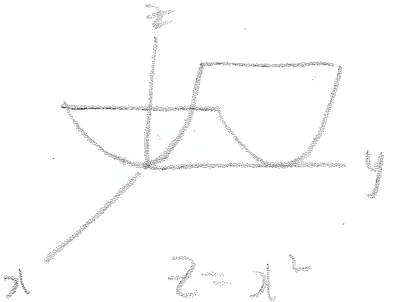
$$= \frac{1}{2} \left[\frac{15}{2}\theta - 8\cos\theta - \frac{\sin 2\theta}{4} \right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2} \left[\frac{15\pi}{2} + 8 - \left(\frac{15\pi}{4}\right) \right]$$

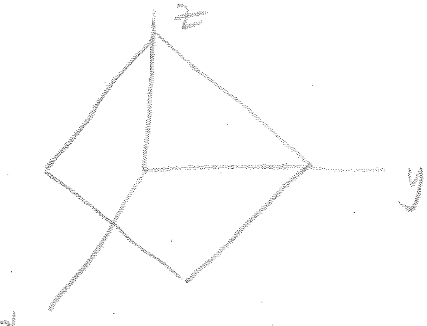
$$= \frac{1}{2} \left[\frac{15\pi}{4} + 8 \right]$$

$$= \frac{15\pi}{8} + 4$$

3. [4 marks] Set up a triple integral for the volume of the solid bounded by $z = x^2$, $y + z = 36$, and $y = 0$. Do not evaluate the integral.



$z = x^2$

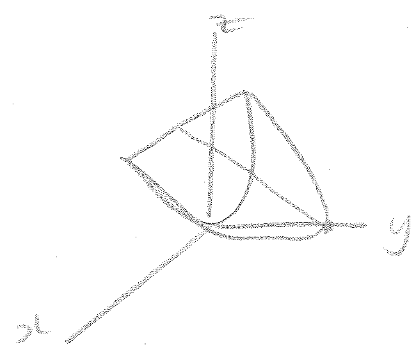


$y + z = 36$

$z = 36 - y$

$\boxed{y = 0 \text{ is the } xz\text{-plane}}$

The solid:



Slice in the z-direction:

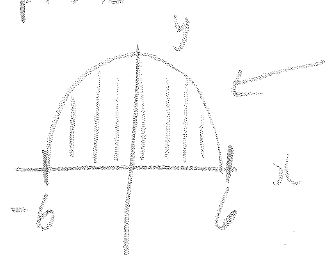
$$x^2 \le z \le 36 - y$$

Project on the xy-plane:

$z = z$

$x^2 = 36 - y$

$y = 36 - x^2$



$y = 36 - x^2$

$$R: \quad 0 \le y \le 36 - x^2$$
$$-6 \le x \le 6$$

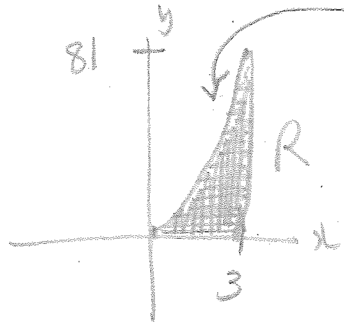
$$V = \int_{-6}^{6} \int_{0}^{36-x^2} \int_{x^2}^{36-y} dz\, dy\, dx$$

4. [3 marks] Rewrite the integral using vertical slices instead of horizontal slices. Do not evaluate.

$$\int_0^{81} \int_{\sqrt[4]{y}}^{3} \sqrt{1+x^5}\, dx\, dy$$

R: $\sqrt[4]{y} \le x \le 3$

$0 \le y \le 81$



$x = \sqrt[4]{y}$

$x^4 = y$

$y = x^4$

R: $0 \le y \le x^4$

$0 \le x \le 3$

Integral $= \displaystyle\int_0^3 \int_0^{x^4} \sqrt{1+x^5}\, dy\, dx$

5. [6 marks] Given $\overbrace{2x+5y+3z}^{g} = 26$. Use the Lagrange Multiplier method to find the point (x, y, z) at which $f = (x-4)^2+(y+2)^2+(z-3)^2$ is minimized.

$$\nabla f = \lambda \nabla g$$

$$[2(x-4), 2(y+2), 2(z-3)] = \lambda [2, 5, 3]$$

$$2(x-4) = 2\lambda \implies \lambda = x-4$$

$$2(y+2) = 5\lambda \implies \lambda = \frac{2}{5}(y+2)$$

$$2(z-3) = 3\lambda \implies \lambda = \frac{2}{3}(z-3)$$

$$\lambda = \lambda = \lambda$$

$$x-4 = \frac{2}{5}(y+2) = \frac{2}{3}(z-3)$$

Get both y and z in terms of x

$$\frac{5}{2}(x-4) = y+2 \qquad \frac{3}{2}(x-4) = z-3$$

$$y = \frac{5}{2}(x-4)-2 \qquad z = \frac{3}{2}(x-4)+3$$

$$y = \frac{5}{2}x - 12 \qquad z = \frac{3}{2}x - 3$$

$$\begin{matrix} y = \frac{5}{2}x - 12 \\ z = \frac{3}{2}x - 3 \end{matrix} \longrightarrow 2x + 5y + 3z = 26$$

$$2x + \frac{25}{2}x - 60 + \frac{9}{2}x - 9 = 26$$

$$19x = 95$$

$$x = 5$$

$$\implies y = \frac{5}{2}x - 12 = \frac{1}{2}$$

and $z = \frac{3}{2}x - 3 = \frac{9}{2}$

f is minimized at $(x, y, z) = (5, \frac{1}{2}, \frac{9}{2})$