1. [4 marks] Calculate the divergence and curl of $\mathbf{F} = [x^2 - yz, e^y - xz, z^3 - y^2]$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(e^y - xz) + \frac{\partial}{\partial z}(z^3 - y^2)$$

$$= 2x + e^y + 3z^2$$

$$\operatorname{Curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & e^y - xz & z^3 - y^2 \end{vmatrix}$$

$$= \vec{i}\,[-2y + x] - \vec{j}\,[0 + y] + \vec{k}\,[-z + z]$$

$$= [x - 2y, -y, 0]$$

2. [4 marks] Evaluate $\int_C (2x + 3z)\, ds$ where C is given by:

$x = 3\cos t, \quad y = 3\sin t, \quad z = 4t, \quad 0 \le t \le 4\pi$

$$\frac{dx}{dt} = -3\sin t \qquad \frac{dy}{dt} = 3\cos t \qquad \frac{dz}{dt} = 4$$

$$ds = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}\, dt$$
$$= \sqrt{9\sin^2 t + 9\cos^2 t + 4^2}\, dt$$
$$= \sqrt{9 + 16}\, dt$$
$$= \sqrt{25}\, dt$$
$$= 5\, dt$$

$$\int_C (2x + 3z)\, ds = 5\int_0^{4\pi} (6\cos t + 12t)\, dt$$
$$= 5\left[6\sin t + 6t^2\right]_0^{4\pi}$$
$$= 5[96\pi^2]$$
$$= 480\pi^2$$

3. [3 marks] Consider the conservative vector field $\mathbf{F} = [2x, 3y^2z, y^3 - 2z]$.

a) Find a potential for $\mathbf{F}$

$f = \int 2x\,dx$ AND $f = \int 3y^2 z\,dy$ AND $f = \int (y^3 - 2z)\,dz$

$= x^2 + g(y,z)$ $\quad = y^3 z + h(x,z)$ $\quad = y^3 z - z^2 + k(x,y)$

$$\Rightarrow f = x^2 + y^3 z - z^2$$

b) Let C be any path from $(1, 2, 3)$ to $(4, 3, 5)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$

Because $\vec{F}$ is conservative:

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$, where f is a potential for $\vec{F}$

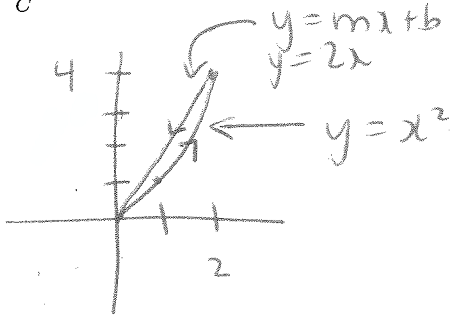
$$= f \Big|_A^B$$

$$= (x^2 + y^3 z - z^2) \Big|_{(1,2,3)}^{(4,3,5)}$$

$$= 126 - 16$$

$$= 110$$

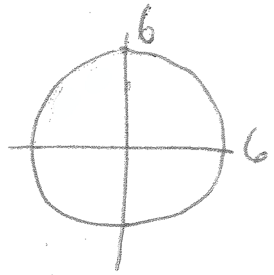
4. [5 marks] Let C be the part of $y = x^2$ from $(0,0)$ to $(2,4)$ followed by the straight line segment from $(2,4)$ to $(0,0)$. Use Green's Theorem to evaluate $\oint_C [\arctan e^x \, dx + (x^2 + \arcsin e^y) dy]$



$$\boxed{R: \quad \begin{array}{c} x^2 \le y \le 2x \\ 0 \le x \le 2 \end{array}}$$

$$\begin{aligned}
\oint (P\,dx + Q\,dy) &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \int_0^2 \int_{x^2}^{2x} 2x \, dy\, dx \\
&= \int_0^2 2xy \Big|_{y=x^2}^{y=2x} dx \\
&= \int_0^2 (4x^2 - 2x^3)\, dx \\
&= \left[\frac{4x^3}{3} - \frac{x^4}{2} \right]_0^2 \\
&= \frac{32}{3} - 8 \\
&= \frac{8}{3}
\end{aligned}$$

5. [4 marks] Use the 2D Divergence Theorem to calculate the flux of $\mathbf{F} = [xy^2, x^2y + \ln(\sin x)]$ across the circle of radius 6 centred at the origin.



Use polar

$$\boxed{R: \quad \begin{array}{l} 0 \le r \le 6 \\ 0 \le \theta \le 2\pi \end{array}}$$

$$\Phi = \iint_R (\operatorname{div} \vec{F})\, dA$$

$$= \iint_R (\underbrace{y^2 + x^2}_{r^2})\, \underbrace{dA}_{r\,dr\,d\theta}$$

$$= \int_0^{2\pi}\int_0^6 r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^6\, d\theta$$

$$= \frac{6^4}{4}\, \cancel{\int_0^{2\pi} d\theta}\ 2\pi$$

$$= 648\pi$$