

Math 109 Skeleton Notes

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Course Overview

Finite Math is a collection of real-world applications that build on Math 11. Here are the main topics, with an example application for each topic.

Chapters 3-5 Sets, Counting and Probability

Is an event likely or unlikely to happen?

Chapters 1-2 Linear Programming

How to maximize profit with a fixed amount of capital and raw materials?

Chapters 6-7 Matrices and Markov Chains

Predict a company's marketshare two years from now.

Chapter 8 Financial Math

What will your monthly payment be when taking out a loan?

Chapter 9 Logic

Helpful when reading legal documents like employment contracts or tenancy agreements.

Chapter 3: Sets and Counting

3.1 Sets

Definition: A **set** is a collection of objects. The objects are called **elements**.

Here is a set with three elements:

$$A = \{x, y, z\}$$

Here is a set with four elements:

$$B = \{3, 5, 7, 9\}$$

Example: Write down the following sets:

a) $A = \{\text{all the letters before f in the English alphabet}\}$

b) $B = \{\text{all the even numbers between 5 and 9}\}$

c) $C = \{\text{all the even numbers between 5 and 7}\}$

Definition: The **union** of sets A and B is the set of elements that are in A or B or both. It is written $A \cup B$.

Definition: The **intersection** of sets A and B is the set of elements that are in both A and B . It is written $A \cap B$.

Example: Let $A = \{-3, -1, 3\}$ and $B = \{-2, -1, 3\}$. Find:

a) $A \cup B$

b) $A \cap B$

Example: Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$ and $C = \{0, 1, 5, 6\}$. Find $(A \cup B) \cap C$.

Definition: B is a **subset of A**, written $B \subseteq A$, if every element of B is an element of A.

Example: Let's write down some examples of subsets.

Definition: The **empty set** contains no elements. It is written $\emptyset$.

Fact: The empty set is a subset of every set.

Example: Let's write down some examples involving the empty set.

Example: List all the subsets of $\{x, y, z\}$.

Definition: The **complement of A** is the set of elements in the universal set U that are not in A. It is written A' .

Example: Let $U = \{a, b, c, d, f\}$, $A = \{b, c\}$ and $B = \{c, d, f\}$. Find:

a) A'

b) $(A \cup B)'$

c) $A' \cup B$

Notation: $b \in A$ means: b is an element of set A .

Example: Let's write down some examples to practice notation for elements and subsets.

3.2 The Inclusion-Exclusion Principle

Notation: $n(S)$ means the number of elements in set S .

Example: Let $A = \{a, b, c\}$ and $B = \emptyset$. Find $n(A)$ and $n(B)$.

Fact: The Inclusion-Exclusion Principle.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Example: Confirm the Inclusion-Exclusion Principle for $A = \{w, x, y\}$ and $B = \{x, y, z\}$.

Definition: A **Venn diagram** is a way to visualize different sets.

Example: Let's draw Venn diagrams for A , B , $A \cap B$, $A \cup B$, A' and B' .

Example: Let $A = \{a, b, c, d, e\}$ and $B = \{b, d, f\}$. Draw a Venn diagram for A and B showing the elements. Draw another Venn diagram for A and B showing the number of elements. Then confirm the Inclusion-Exclusion Principle for A and B .

Example: A company has 300 employees: 275 are full-time and 230 are permanent, while 285 are full-time or permanent. How many are full-time and permanent?

Example: Draw a Venn diagram for $A \cap B'$.

Example: Draw a Venn diagram for $(A \cup B) \cap C'$.

Fact: De Morgan's Laws.

For any sets S and T :

$$(S \cup T)' = S' \cap T'$$

$$(S \cap T)' = S' \cup T'$$

Example: Use Venn diagrams to confirm that $(S \cap T)' = S' \cup T'$.

Example: Simplify $(A \cup B)'$.

Comment: The last example shows that there can be multiple ways to describe a given set.

3.3 Venn Diagrams and Counting

A Venn diagram for two sets contains four **basic regions**. Let's draw a Venn diagram for sets A and B , and observe the four basic regions.

Example: We are given: $n(U) = 60, n(A) = 18, n(B) = 22$ and $n(A \cap B) = 15$.

a) Draw a Venn diagram.

b) How many elements are in A but not in B ?

Example: We are given: $n(U) = 120$, $n(A) = 48$, $n(B) = 52$ and $n(A \cup B) = 90$. Draw a Venn diagram.

Example: Out of 150 students, 60 are first-year, 112 have a job and 20 are non-first-year with no job. Draw a Venn diagram.

A Venn diagram for three sets contains eight **basic regions**. Let's draw a Venn diagram for sets A , B and C , and observe the eight basic regions.

Example: Of 100 employees: 65 speak French, 55 speak Spanish, 52 speak Arabic, 34 speak French and Spanish, 29 speak French and Arabic, 20 speak Spanish and Arabic, and 3 speak all three languages. Draw a Venn diagram.

Example Continued...

Example: Each of 100 job applicants has at least one type of experience: presentation, coding or international experience. Draw a Venn diagram given that:

45 have presentation experience

56 have coding experience

41 have international experience

11 have presentation and coding experience

20 have presentation and international experience

18 have coding and international experience

Example Continued...

Example Continued...

3.4 The Multiplication Principle

In this section and the next two sections we'll look at how many ways there are to perform a certain activity. This will prepare us for the concept of probability in Chapter 4.

Fact: The Multiplication Principle

When performing a sequence of tasks, the number of possibilities is multiplied.

Example: There are six different routes from City A to City B, and two different routes from City B to City C. How many possible routes are there from City A to City C?

Example: How many possible outcomes are there if we toss a coin four times?

Example: How many six-digit palindromes are there?

Example: Count the number of five-digit passcodes using digits 0 to 9 if adjacent digits can't be the same.

Example: How many ten-digit phone numbers have area codes 250, 778 or 236 such that the last seven digits aren't 000-0000?

Example: How many ways are there to arrange seven books on a shelf from left to right?

Example: Ten people interview for a job.

a) How many ways are there to choose the best and second-best candidate?

b) How many ways are there to rank all ten people from 1st to 10th?

Example: How many three-letter “words” (including nonsense words) can be formed from A, B, C if:

a) repetition is allowed

b) repetition is not allowed

3.5 Permutations and Combinations

Definition: A **permutation** is an ordered selection of r objects from a group of n objects.

Notation: The number of permutations is written $P(n, r)$.

Fact: $P(n, r) = n \times (n - 1) \times (n - 2) \times \cdots \times (n - r + 1)$

Example: Compute the following in two ways: using the formula above and using your calculator.

a) $P(40, 3)$

b) $P(4, 4)$

c) $P(8, 1)$

Example: How many two-letter “words” (including nonsense words) can be formed from A,B,C,D if repetition is not allowed?

Notation: $n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$
It is pronounced “n factorial.”

Example: Compute the following in two ways: using the formula above and using your calculator.

a) $5!$

b) $3!$

c) $1!$

Comment: Note that $0! = 1$ by definition. You can confirm this on your calculator.

Example: How many ways are there to arrange four books in a row?

Fact: $P(k, k) = k!$ for $k = 0, 1, 2, \dots$

Definition: A **combination** is an unordered selection of r objects from a group of n objects.

Notation: The number of combinations is written $C(n, r)$.

Fact: $C(n, r) = \frac{P(n, r)}{r!}$

Example: Compute the following using your calculator.

a) $C(40, 3)$

b) $C(4, 4)$

c) $C(8, 1)$

Example: How many ways are there to select two people from a group of four?

Example: We have interviewed 20 candidates for a job. How many ways are there to select our 1st, 2nd and 3rd choice?

Example: A class has 45 students. How many ways are there to form a four-person team?

Example: In a batch of 150 numbered phones, four are defective.

a) How many ways are there to select three phones from the batch?

b) How many ways are there to select three defective phones from the batch?

The standard deck of cards sometimes comes up in counting problems.

Example: In this example we'll write down everything we need to know about the standard deck of cards.

Example: How many five-card hands from a standard deck have:

a) only hearts?

b) no hearts?

Example: How many ways are there to select four of seven books and arrange them in a row?

Example: Five students are giving presentations. How many orders are possible if Al goes last and Bob goes first or second?

Example: How many ways are there to arrange four pairs of people in a row so that each pair is adjacent?

3.6 Further Counting Techniques

Example: A box contains five red numbered balls and four yellow numbered balls. Three balls are selected.

a) How many possible selections are there?

b) How many selections consist of two red balls and one yellow ball?

c) How many selections consist of at least two red balls?

Example: A full house in poker consists of three cards of one denomination and two cards of a different denomination, for example QQQ88 or 222KK. How many possible full house hands are there?

Example: We toss a coin four times. How many ways are there to get exactly two heads?

Example: We toss a coin six times. How many outcomes have:

a) at most two heads?

b) at least two heads?

Example: A city has all its streets running north-south or east-west to form a rectangular grid. Corner B is five blocks east and four blocks south of Corner A. How many shortest routes are there from Corner A to Corner B?

Example: A city has all its streets running north-south or east-west to form a rectangular grid. Corner B is five blocks east and four blocks south of Corner A. How many shortest routes are there from Corner A to Corner B that don't go south for two consecutive blocks?

Example: Five students sit in a row. How many arrangements are there so that Al is to the right of Bob?

Example: How many ways are there to arrange the letters A, B, C, D, E, F if the letters D, E, F must be beside one another in some order?

Chapter 4: Probability

4.1 Sample Space and Events

In this chapter we'll consider whether specific events are likely or unlikely.

Definition: The **sample space** is the set of possible outcomes of an experiment. An **event** is a subset of the sample space.

Example: Suppose we roll a (six-sided) die. Let's write out the sample space and some possible events.

Example: We toss a coin three times and record heads or tails.

a) Write out the sample space.

b) Let E : at most one tail. Write out E .

Example: A small team has four employees named Al, Bo, Cindy and Dan. We want to select two of them for a project.

a) Write out the sample space.

b) Let E: Al is not chosen. Write out E.

Example: Suppose an experiment has sample space $S = \{1, 2\}$. Write out all the possible events.

Example: We roll a red die and a blue die.

a) How many outcomes are there in the sample space?

b) Let E : the rolls sum to six. Write out E .

Example: We roll a red die and a blue die.

Let F : the roll on the red die is 4.

Let G : the roll on the red die is 3 or 4, and the roll on the blue die is at least 5.

a) Write out F .

b) Write out G .

c) Find $F \cup G$.

d) Find $F' \cap G$.

Definition: Two events E and F are **mutually exclusive** if $E \cap F = \emptyset$.

Comment: Here are two ways of rephrasing “ E and F are mutually exclusive”:

“ E and F have no outcomes in common”

“ E and F can’t happen at the same time”

Example: We flip a coin three times. Are the following events mutually exclusive?

a) E : No heads appear.

F : No tails appear.

b) E : At most one tail appears.

F : No tails appear.

4.2 Basic Probability Concepts

Definition: The **probability** of an event is a measure of how likely the event is.

Notation: We often write probabilities as decimals. $Pr(E) = 0.35$ means that the probability of event E is 35%.

Fact: Any probability is always between 0 and 1 inclusive.

$Pr(F) = 0$ means the probability of event F is 0%, in other words event F cannot happen.

$Pr(G) = 1$ means the probability of event G is 100%, in other words event G is guaranteed to happen.

Fact: For any experiment, if we sum the probabilities of all the outcomes we get 1.

Definition: A **probability distribution** is a table that lists the different outcomes of an experiment and their probabilities.

Example: We toss a fair coin and record heads or tails. Let's write down the probability distribution.

Example: Some students are polled on their program. Use the following information to find the probability distribution.

Program	Number of Students
Business	13
Technology	18
Nursing	9

Fact: The probability of an event is the sum of the probabilities of the relevant outcomes.

Example: An unfair four-sided die has the following probability distribution:

Roll	Probability
1	0.1
2	0.35
3	0.3
4	0.25

a) Find the probability that a roll is less than 3.

b) Find the probability that a roll is odd.

c) Find the probability that a roll is less than 2 and even.

Example: An experiment has possible outcomes A, B and C. We are given $Pr(A) = 0.4$ and we are told that outcome B is three times as likely as outcome C. Find the probability distribution.

Fact: Inclusion-Exclusion Principle

$$Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$$

Comment: Compare this with Section 3.2:

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

Example: Given $Pr(E) = 0.6$, $Pr(F) = 0.5$ and $Pr(E \cup F) = 0.65$.

a) Find $Pr(E \cap F)$

b) Draw a Venn diagram

c) Find $Pr(E' \cap F)$

Fact: Complement Rule

$$Pr(E) = 1 - Pr(E')$$

Comment: This is true because $Pr(E) + Pr(E') = 1$.

The Complement Rule can also be rephrased as: $Pr(E') = 1 - Pr(E)$

Example: A company has two suppliers. The probability that Supplier 1 is late is 20%. The probability that Supplier 2 is late is 15%. The probability that both suppliers are late is 8%. Find the probability that neither supplier is late.

Example Continued...

4.3 Calculating Probabilities

Fact: When all the outcomes of an experiment are equally likely, the probability of an event E is:

$$Pr(E) = \frac{n(E)}{n(S)}$$

Example: A box contains four defective and five good items. We randomly select three items from the box. Find the probability that:

a) no defective items are selected

b) two defective items are selected

c) at least two defective items are selected

Example: A fair die is rolled seven times. Find the probability of rolling exactly five 3's.

Example: A five-card poker hand is dealt. Find the probability of getting two pairs. For example, the hand could be 88QQK or 33772.

Recall the Complement Rule:

$$Pr(E) = 1 - Pr(E')$$

Comment: This is true because $Pr(E) + Pr(E') = 1$.

The Complement Rule can also be rephrased as: $Pr(E') = 1 - Pr(E)$

Example: In a class of 40 students, six have the flu. Five students are randomly selected from the class. Find the probability that at least one of them has the flu.

Example: A group of 20 people are randomly selected. Find the probability that at least two of them are born on the same day of the year. We'll assume 365 days in a year (meaning ignore February 29).

4.4 Conditional Probability and Independence

Notation: The probability of event E, given that event F occurs is written $Pr(E|F)$. We pronounce this “the probability of E, given F.”

Fact: $Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$

Comment: If all the outcomes of an experiment are equally likely, then there is another version of this formula that is also true:

$$Pr(E|F) = \frac{n(E \cap F)}{n(F)}$$

Example: Some university graduates are randomly polled on the degree they received and whether they are currently employed. The number of graduates in each category is below.

	Employed	Unemployed
Arts	87	13
Business	198	2

a) Find the probability that a graduate is employed, given that they majored in business.

b) Write this information in a sentence.

c) Find the probability that a graduate got a business degree, given that they're employed.

d) Write this information in a sentence.

Example: A fair coin is tossed three times. Let:

E: three heads appear

F: at least two heads appear

Find the probability of E, given F.

Example: At a certain company:

60% of employees are college graduates

50% of employees are permanent

20% of employees are permanent and college graduates

Find the probability that an employee who is permanent is a college graduate.

Example: At a certain company:

40% of employees are permanent

30% of permanent employees have a pension

Find the probability that an employee is permanent and has a pension.

Definition: Events E and F are **independent** if the occurrence of one of the events does not change the probability of the other.

Definition: If two events are not independent then we call them **dependent**.

Example: Here are some examples of **independent** events:

- a) the first toss of a coin is heads and the second toss is heads
- b) the first roll of a die is odd and the second roll is odd

Example: Here are some examples of **dependent** events:

- a) it rains today and it rains tomorrow
- b) a company's stock price increases today and it increases tomorrow

Fact: The following statements are equivalent. If one statement is true, then the other two statements are true. If one statement is false, then the other two statements are false.

- 1) Events E and F are independent
- 2) $Pr(E|F) = Pr(E)$
- 3) $Pr(E \cap F) = Pr(E) \cdot Pr(F)$

Example: We are given: $Pr(E) = 0.4$, $Pr(F) = 0.3$ and $Pr(E \cup F) = 0.5$
Are E and F independent?

Example: We roll a fair die. Let $E = \{2, 3, 5\}$ and $F = \{1, 2\}$.
Are E and F independent?

Example: We are given: $Pr(E) = 0.6$ and $Pr(F) = 0.3$
Find $Pr(E \cap F)$, $Pr(E|F)$, and $Pr(F|E)$ if:

a) E and F are independent

b) E and F are dependent

Example: A plane has two independent engines. Engine 1 fails on 3% of flights.
Engine 2 fails on 2% of flights. Find the probability that at least one engine works properly on the next flight.

Fact: Three events E, F and G are independent exactly when:

$$Pr(E \cap F \cap G) = Pr(E) \cdot Pr(F) \cdot Pr(G)$$

Comment: Similarly, more than three events are independent exactly when: the probability of their intersection is equal to the product of their probabilities

Example: Write down a formula that represents “events E, F, G and H are independent.”

Example: A product has three independent parts. In a year of use:

Part A fails 1% of the time

Part B fails 2% of the time

Part C fails 4% of the time

Find the probability that no part fails in a year of use.

Example: A shipment contains nine good and two defective items. Items are selected one at a time (without replacement) until a good item is found.

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Example: At a college:

$\frac{3}{5}$ of students are in Business

$\frac{2}{5}$ of students are in Technology

$\frac{1}{2}$ of Business students have a job

$\frac{1}{3}$ of Technology students have a job

Find the probability that a student:

a) is in Business and has a job

b) has a job

c) is in Business, given that they have a job

Example: 30% of days are rainy.
80% of rainy days are windy.
10% of non-rainy days are windy.
Find the probability that a windy day is rainy.

Example: Consider the context of testing a person for a disease (for example, testing to see if they have the flu). Draw the relevant tree diagram.

Comment: Tests can give an incorrect result due to human error, equipment error etc.

Example: 5% of patients have a certain disease.

The false-positive rate is 2%.

The false-negative rate is 3%.

Draw a tree diagram.

Example: 3% of students have the flu.

Of people with the flu, 96% test positive.

Of people not having the flu, 95% test negative.

Find the probability that a person who tests positive actually has the flu.

Comment: This probability is surprisingly low. When a disease is rare in the population, many of the positive results can be false positives.

Example: 30% of students have the flu.

Of people with the flu, 96% test positive.

Of people not having the flu, 95% test negative.

Find the probability that a person who tests positive actually has the flu.

Comment: We've used tree diagrams to solve the problems in this section. There is a formula called **Bayes' Theorem** that captures the same idea using algebra rather than tree diagrams.

4.6 Bigger Tree Diagrams

Example: A factory has four production lines.

Line 1 produces 15% of all items; 2% of the items from Line 1 are defective.

Line 2 produces 40% of all items; 3% of the items from Line 2 are defective.

Line 3 produces 35% of all items; 7% of the items from Line 3 are defective.

Line 4 produces 10% of all items; 4% of the items from Line 4 are defective.

Find the probability that a defective item was produced on Line 3.

Chapter 5: Expected Value and Binomial Experiments

5.1 Expected Value

In this chapter we'll look at two specific probability concepts that come up in business, science, and social science.

Notation: In Chapter 5 we'll write probabilities as $P(E)$ instead of $\Pr(E)$. Feel free to use either notation.

Notation: In Chapter 5 we'll write combinations in the format $5C3$ instead of $C(5, 3)$. Feel free to use either notation.

Definition: A **random variable** assigns a number to each outcome of an experiment. Random variables are written X .

Example: Three fair coins are flipped. Let X be the number of heads that appear. Find the probability distribution of X .

Example: Referring to the previous example, let's draw the histogram (bar chart) that represents X .

Definition: The **expected value** of a random variable X is the theoretical average of X if the experiment were repeated infinitely-many times. It is sometimes called the **mean of X** .

Notation: The expected value of X is written μ or $E(X)$. Feel free to use either notation.

Fact: $\mu = x_1p_1 + x_2p_2 + \dots + x_np_n$ where:

$x_1, x_2, \dots, x_n$ are the values of X

$p_1, p_2, \dots, p_n$ are their respective probabilities

Example: A fair die is rolled. Let X be the number rolled. Find the expected value of X .

Example: A box contains nine \$5 bills and six \$10 bills. You pay \$8 and randomly draw a bill from the box. Let X be your net winnings (in dollars). Find the expected value of X .

Example: You insure a used car worth \$4,000 against theft for one year by paying a premium of \$112. The probability that the car is stolen during this year is 1.3%. Find your expected net gain (in dollars) on the insurance policy.

Example: A shipment contains ten good and five defective items. We randomly select three items from the shipment. Find the expected number of good items that are selected.

Definition: A game is **fair** if the expected net winnings is equal to zero.

Example: You pay \$1 to roll a fair die. If you roll a 1 or a 6, you win \$5. Otherwise, you must pay k more dollars. Find k so that the game is fair.

5.2 Binomial Experiments

Definition: A **Bernoulli trial** is an experiment with only two outcomes, usually called “success” and “failure.”

Notation: Let p be the probability of success on a Bernoulli trial.
Let q be the probability of failure on a Bernoulli trial.

Fact: For any Bernoulli trial, $q = 1 - p$. This is true because $p + q = 1$.

Example: We roll a fair die. Suppose a success is rolling a 4. Calculate p and q .

Definition: A **binomial experiment** is a sequence of n independent Bernoulli trials.

Example: We roll a fair die seven times. Let a success be rolling less than 3.
Calculate n , p and q .

Fact: The probability of exactly x successes in a binomial experiment is:

$$(nC_x)p^xq^{n-x}$$

where n is the number of trials,

p is the probability of success on one trial, and

q is the probability of failure on one trial.

b) the probability of rolling more than three 2's.

c) the probability of rolling fewer than four 2's.

Example: A basketball player makes 65% of his free throws. He takes three shots and he does not improve with practice.

a) Let X be the number of successful free throws the player makes. Find the probability distribution of X .

b) Draw a histogram (bar chart).

Recall from Section 5.1:

- The expected value of a random variable X is the theoretical average of X if the experiment were repeated infinitely-many times. It is sometimes called the **mean of X** .
- The expected value of X is written μ or $E(X)$. Feel free to use either notation.
- $\mu = x_1p_1 + x_2p_2 + \dots + x_np_n$ where:
 $x_1, x_2, \dots, x_n$ are the values of X
 $p_1, p_2, \dots, p_n$ are their respective probabilities

Fact: Let X be the number of successes in a binomial experiment. Then $\mu = np$.

Example: Thirty percent of households in Victoria have pets. We select eight households at random.

a) Find the probability that at most two households in the sample have pets.

b) Find the probability that at least two households in the sample have pets.

c) Find the expected number (or mean number) of households in the sample that have pets.

Chapter 1: Lines and Inequalities

1.1 Lines

In this chapter we'll learn how to solve equations and inequalities. This will lead us into solving more applied problems in Chapter 2.

Example: Plot the points $A = (2, 3)$, $B = (-2, 3)$, $C = (-2, -3)$, and $D = (2, -3)$.

Example: Consider the line $21x + 7y = 14$.

a) Put the equation into the standard form $y = mx + b$.

b) What is the slope of the line?

c) What is the y -intercept of the line?

d) What is the x -intercept of the line?

e) Is the point $(1, -1)$ on the line?

f) Is the point $(2, 1)$ on the line?

Example: Graph $2x + 3y = 12$.

Example: Graph $4x + y = 0$.

Example: a) Write the equation of a horizontal line. Graph it.

b) Write the equation of a vertical line. Graph it.

Example: Annual car insurance for a certain model of car in a small town costs $y = 65x + 712$, where y is in dollars and x is the number of years after 2010.

a) What was the cost in 2015?

b) In which year was the cost \$1167?

c) Interpret the y -intercept.

1.2 Linear Inequalities

Example: Let's write down some inequalities:

Example: Let's draw the number line to help visualize these inequalities:

Fact: We can add any number to both sides of an inequality.
We can subtract any number from both sides of an inequality.

Fact: We can multiply both sides of an inequality by any nonzero number,
but **multiplying by a negative number reverses the inequality**.

Fact: We can divide both sides of an inequality by any nonzero number,
but **dividing by a negative number reverses the inequality**.

Example: Solve $6 - 2x \geq 8$.

Definition: The **standard form** for an inequality is: $y \leq mx + b$ or $y \geq mx + b$.

Example: a) Put $8x - 4y \geq 12$ in standard form.

b) Does $(0, 0)$ satisfy $y \leq 2x - 3$?

c) Does $(2, -1)$ satisfy $y \leq 2x - 3$?

Example: Put $0.3x - 0.4y \leq 2$ in standard form.

Example: Put $\frac{2}{7}x - 3y \leq \frac{4}{7}$ in standard form.

Fact: To graph an inequality, we first graph the associated line. All the points that satisfy the inequality will be on one side of the line. To figure out which side, we test any point that is not on the line.

Example: Graph $8x - 4y \geq 12$.

Example: Graph $-3x + y \geq 0$.

Example: Graph the feasible set for the system of inequalities:

$$\begin{aligned}x + 1 &\geq 0 \\x + y &\leq 10 \\-x + 0.5y &\geq 0.5\end{aligned}$$

Example Continued...

Example: Graph the feasible set for the system of inequalities:

$$\begin{aligned}x + y &\leq 6 \\ -2x + y &\geq 0 \\ y &\geq 2 \\ x &\geq 0\end{aligned}$$

Example Continued...

1.3 Intersection of Two Lines

Example: Find the intersection of the lines below. In other words, solve the system of equations.

$$3x + 2y = 2$$

$$9x - 2y = 18$$

Example: Find the intersection of the lines below. In other words, solve the system of equations.

$$x = -1$$

$$y = -3x + 2$$

Example: Graph the lines $x = 0$, $-2x + y = 2$ and $4x + 2y = 12$. Find the vertices of the triangle formed by the three lines.

Example Continued...

Let's look at the concept of supply and demand in business. We will graph the price of an item (written p) as a function of the quantity (written q).

Example: Draw the quantity axis and the price axis.

Definition: The supply curve is a line. It shows the relationship between the price of an item and the quantity that producers are willing to make.

Definition: The demand curve is a line. It shows the relationship between the price of an item and the quantity that consumers are willing to purchase.

Definition: The equilibrium point is the point where the supply curve and the demand curve intersect.

Example: Draw a typical supply curve, a typical demand curve, and the equilibrium point.

Example: We are given a supply curve and a demand curve. The price p is in dollars and the quantity q is in thousands of units. Find the equilibrium quantity and price.

$$p = 0.005q + 2.5$$

$$p = -0.002q + 6.7$$

Fact: A cost line has the form $y = mx + b$, where y is the cost in dollars and x is the number of units produced. The number m is called the **marginal cost** or the **cost per unit**, and the number b is called the **fixed cost**.

Example: Graph a typical cost line.

Example: Manufacturer A can produce a product for \$300 plus \$10 per unit. Manufacturer B can produce a product for \$200 plus \$12 per unit. How many units result in the costs being equal? What is the cost for this number of units?

Example Continued...

1.4 Slope of a Line

Definition: The **standard form** of a line is $y = mx + b$. This is also called the **slope-intercept form**. The number m is the slope of the line and the number b is the y -intercept.

Fact: Steepness Property

$$m = \frac{\text{rise}}{\text{run}}$$

Example: Graph the line $y = -\frac{3}{4}x + 2$ using the slope and y -intercept.

Fact: The slope of the line through (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the slope of the line through the points $(-1, 4)$ and $(2, 10)$.

Example: Find the equation of the line with a slope of $\frac{4}{5}$ that passes through $(-1, 2)$.

Example: Find the equation of the line through the points $(2, 7)$ and $(-3, 15)$.

Fact: Parallel lines have the same slope.

Fact: Suppose Line 1 and Line 2 are perpendicular to one another. If Line 1 has slope m_1 and Line 2 has slope m_2 , then $m_2 = -\frac{1}{m_1}$.

Example: Find the equation of the line passing through $(-2, 5)$ that is:

a) parallel to $y = -\frac{4}{3}x + 1$

b) perpendicular to $y = -\frac{4}{3}x + 1$

Recall from Section 1.3:

A cost line has the form $y = mx + b$, where y is the cost in dollars and x is the number of units produced. The number m is called the **marginal cost** or the **cost per unit**, and the number b is called the **fixed cost**.

Example: Graph a typical cost line. What does the slope represent? What does the y -intercept represent?

Example: The cost of making 300 laptops is \$126,500. The cost of making 750 laptops is \$306,500. Find the equation of the cost line.

Example Continued...

Chapter 2: Linear Programming

2.1 and 2.2 Linear Programming

Suppose a manufacturing company has a fixed amount of raw materials and wants to maximize their profit. In this chapter we'll use our understanding of equations and inequalities to maximize or minimize a quantity with some restrictions. This process is called **linear programming**.

Fact: The Fundamental Theorem

The maximum (or minimum) value of the objective function occurs at one of the vertices of the feasible set.

Comment: Each linear programming problem will be divided into eight short steps. We'll revisit the fact above during the seventh step.

Example: Each day a company has 60kg of wood and 100kg of metal available. A chair uses 2kg of wood, 4kg of metal and yields a profit of \$14. A table uses 3kg of wood, 4kg of metal and yields a profit of \$20. How many chairs and tables maximize the daily profit?

1) Variables

2) Chart

3) Inequalities

4) Graph Feasible Set

5) Find all Vertices

6) Objective Function

7) Table

8) Answer

Example: Astronauts have two foods: Food Alpha and Food Beta. Food Alpha has 12g of fat and 50g of carbs per serving and has a mass of 0.4kg per serving. Food Beta has 15g of fat and 20g of carbs per serving and has a mass of 0.3kg per serving. Astronauts require at least 60g of fat and 200g of carbs per day. How many servings of Food Alpha and Food Beta will minimize the total food mass per day?

1) Variables

2) Chart

3) Inequalities

4) Graph Feasible Set

5) Find all Vertices

6) Objective Function

7) Table

8) Answer

Chapter 6: Matrices

6.1 Systems of Equations with Unique Solutions

In this chapter we'll focus on solving systems of equations. This will prepare us for more applied problems in Chapter 7.

Definition: A **matrix** is a rectangular array of numbers. We say “one matrix” and “two or more matrices”.

Example: Write the following system of equations in matrix form:

$$\begin{aligned}9x - 2y &= 18 \\3x + 4y &= 27\end{aligned}$$

Fact: A system of equations can have no solution, one unique solution, or infinitely-many solutions.

Example: Draw a picture to illustrate each of the three scenarios in the fact above.

Definition: A matrix is in **diagonal form** if it looks like $\left[\begin{array}{cc|c} 1 & 0 & \# \\ 0 & 1 & \# \end{array} \right]$ or $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{array} \right]$, where # represents any number.

Fact: Row Operations

To solve a system of equations, we perform operations on the matrix form until it becomes diagonal form (or as close as possible to diagonal form). There are three types of operations we are allowed to perform:

- 1) Swap any two rows
- 2) Multiply or divide any row by a nonzero number
- 3) (Current Row) - #(Pivot Row)

Comment: These operations do not change the solution to the system of equations.

Example: Solve:

$$3x + 6y = 30$$

$$2x + 8y = 32$$

Example Continued...

Comment: This process is called **Gauss-Jordan Elimination**.

Example: Solve using Gauss-Jordan Elimination:

$$\begin{array}{rcrcrcrcrcl} 4x & - & 10y & + & 10z & = & -18 \\ & x & - & 3y & + & 4z & = & -8 \\ -3x & + & 11y & - & 5z & = & 25 \end{array}$$

Example Continued...

Example: Each hat takes 3 hours and \$2 to produce. Each coat takes 6 hours and \$8 to produce. You must spend exactly 30 hours and \$32. How many hats and coats can you produce?

6.2 General Systems of Equations

In this section we'll look at systems of equations with no solution or with infinitely-many solutions.

Example: Solve using Gauss-Jordan Elimination:

$$x + 2y + z = 9$$

$$x + 3y + 3z = 12$$

$$x + 4y + 5z = 1$$

Fact: If you see a row like $\left[\begin{array}{ccc|c} 0 & 0 & & \text{nonzero} \end{array} \right]$ or $\left[\begin{array}{cccc|c} 0 & 0 & 0 & & \text{nonzero} \end{array} \right]$, then the system has no solution.

Example: Solve using Gauss-Jordan Elimination:

$$x + 2y + z = 9$$

$$x + 3y + 3z = 12$$

$$x + 4y + 5z = 15$$

Example: Find three solutions to the system on the previous page.

Example: Solve the following systems. Assume the variables are x and y , in that order.

a) $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array} \right]$

b) $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{array} \right]$

Example: A store sells three items priced at \$7, \$10 and \$13. Amanda wants to buy 15 items in total and she wants to spend exactly \$150. Give three options for what Amanda could buy.

Example Continued...

6.3 Matrix Operations

Over this section and the next two sections we'll learn a second method for solving systems of equations.

Definition: The **size** of a matrix is (number of rows) $\times$ (number of columns).

Example: What is the size of $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$?

Example: Let $\begin{bmatrix} -6 & 2x \\ y+1 & 9 \end{bmatrix} = \begin{bmatrix} -6 & 8 \\ 7 & 9 \end{bmatrix}$. Find x and y .

Example: Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$. Find:

a) $3A$

b) $A + B$

c) $2A - 5B$

Fact: $A + B$ is undefined if A and B have different sizes.

Example: Find $\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \end{bmatrix}$.

Definition: The **dot product** of a row and a column is a number.

Example: Let's calculate some dot products:

a) $\begin{bmatrix} 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 6 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 6 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 3 & -2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$

Fact: To multiply two matrices:

$$AB = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & \dots & r_1 \cdot c_n \\ r_2 \cdot c_1 & r_2 \cdot c_2 & \dots & r_2 \cdot c_n \\ \dots & \dots & \dots & \dots \\ r_n \cdot c_1 & r_n \cdot c_2 & \dots & r_n \cdot c_n \end{bmatrix}$$

where r_i is row i of matrix A and c_j is column j of matrix B .

Example: Find AB where $A = \begin{bmatrix} 4 & 6 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$.

Example: Let's consider the sizes of A and B in the example above. Which two numbers must be equal to make AB defined? Which two numbers predict the size of AB ?

Example: Let A be a 2×3 matrix and let B be a 3×1 matrix. Calculate the sizes of AB and BA .

Fact: $AB \neq BA$ in general.

Example: Find the matrix products CD and DC , if possible.

Given $C = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$.

Definition: The **identity matrix** is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Fact: For any matrix A , $AI = A$ and $IA = A$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Confirm the fact above by finding AI and IA .

Example: Expand the following:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Comment: Matrix multiplication was reverse-engineered to solve systems of equations.

Definition: A **matrix equation** has the form $AX = B$ where:

A contains the coefficients

X contains the variables, written as a column

B contains the constants, written as a column

Example: Write the following as a matrix equation:

$$\begin{aligned} 2x - 9y &= 12 \\ 7x + 3y &= -16 \end{aligned}$$

6.4 The Inverse of a Matrix

Recall from Section 6.3:

The identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Definition: If a matrix A is square (2×2 or 3×3), then the **inverse of A** is written A^{-1} . It has the property $A^{-1}A = I$.

Comment: A^{-1} may be undefined for some square matrices.

Example: Let $A = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$. Check that $\begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix} = A^{-1}$.

Definition: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the **determinant of A** is written D .

It is calculated as: $D = ad - bc$.

Fact: The Inverse of a 2×2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Comment: If $D = 0$, then A^{-1} is undefined.

Example: Find the inverse matrix:

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Fact: If A^{-1} exists, then the system of equations $AX = B$ has the unique solution $X = A^{-1}B$.

Example: Let's explore why this fact is true.

Example: Solve by finding A^{-1} :

$$7x - 3y = 25$$

$$2x + y = 9$$

Example: Use A^{-1} to solve:

$$\begin{array}{rcl} x - 3y & = & -13 \\ 2x + 9y & = & 19 \end{array}$$

6.5 The Inverse of a 3x3 Matrix

Fact: To find A^{-1} for a 3×3 matrix A , we form the matrix $[A|I]$. We perform row operations until we get I on the left side. The matrix on the right side will be A^{-1} .

Comment: By turning A into I , we are “undoing” A . On the right side we are building the matrix that “undoes” A , that is A^{-1} .

Comment: See Section 6.1 to recall the three types of row operations we are allowed to perform.

Example: Find A^{-1} for $A = \begin{bmatrix} 2 & 10 & 2 \\ 0 & 4 & 1 \\ 2 & 14 & 2 \end{bmatrix}$.

Example Continued...

Fact: Suppose a row of zeros appears on the left side while trying to turn $[A|I]$ into $[I|A^{-1}]$. Then A^{-1} is undefined.

Example: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 3 \\ 2 & 2 & 12 \end{bmatrix}$.

Recall from Section 6.4:

If A^{-1} exists, then the system of equations $AX = B$ has the unique solution $X = A^{-1}B$.

Example: Find A^{-1} and use it to solve:

$$5x + 6y + 7z = 56$$

$$x + y + z = 9$$

$$x + \quad \quad z = 6$$

Example Continued...

Chapter 7: Markov Chains

7.1 Markov Chains

In this chapter we'll use matrices to make predictions.

Example: We have the following information about CleanHair Shampoo company:

90% of CleanHair customers will buy CleanHair next time.

10% of CleanHair customers won't buy CleanHair next time.

20% of other brand customers will buy CleanHair next time.

80% of other brand customers won't buy CleanHair next time.

a) Draw the transition diagram.

b) Find the transition matrix.

Fact: Properties of a Transition Matrix:

- 1) It's a square matrix (2×2 or 3×3).
- 2) All entries must be at least zero.
- 3) Each **row** sums to 1.

Comment: In a transition diagram, the arrows **leaving** a state sum to 1.

Definition: The **initial state matrix** is written S_0 .

Example: Interpret $S_0 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$ in the context of the CleanHair Shampoo example.

Fact: Given the initial state matrix S_0 and the transition matrix P :

The next state matrix is $S_1 = S_0P$

The state matrix after that is $S_2 = S_1P$

The state matrix after that is $S_3 = S_2P$

etc.

Example: Suppose $S_0 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$ and $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$, where the states are Y and N in that order. Find S_1 and S_2 and interpret them in the context of the CleanHair shampoo example.

Fact: Given any state matrix $(S_0, S_1, S_2 \text{ etc.})$, the probabilities sum to 1.

Definition: A **Markov chain** is a sequence of state matrices $S_0, S_1, S_2, S_3, \dots$

Example: We have the following information about the price of a certain company's stock:
If the price increases one day then the probability that it increases the next day is 65%.
If the price decreases one day then the probability that it decreases the next day is 45%.

a) Draw the transition diagram.

b) Find the transition matrix.

c) There is an 80% probability that the stock's price will decrease today. Find the probability that the price increases two days from now.

Example Continued...

Comment: Depending on the problem, the next state matrix could represent the next purchase, the next day, the next week, the next year etc.

7.2 The Stationary Matrix

Definition: The **stationary matrix** is written S . It is a state matrix that has the property that $SP = S$.

Fact: If S exists then $S_k \approx S$ for large values of k .

Example: Let $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$.

a) Check that $S = \begin{bmatrix} 0.625 & 0.375 \end{bmatrix}$ and interpret S .

b) What will S_{50} look like?

c) In the distant future, what will the state matrix look like?

Comment: If S exists then the Markov chain is sometimes called a **regular Markov chain**.

Example: Let $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$. Find the stationary matrix.

Example Continued...

Example: Let $P = \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 0.7 \end{bmatrix}$. Find the stationary matrix.

Example Continued...

Chapter 8: Financial Math

8.2 Compound Interest

In this section we'll study one-time investments that earn compound interest.

Definition: An investment earns **compound interest** if the interest is reinvested at the end of each compounding period.

Fact: The formula for a one-time investment earning compound interest is:

$$A = P\left(1 + \frac{r}{m}\right)^{mt}, \text{ where:}$$

A is the future value, in dollars

P is the present value, in dollars (sometimes called the principal)

r is the annual nominal interest rate

m is the number of compounding periods per year

t is the time, in years

Comment: The annual nominal interest rate r is expressed as a decimal.
So 3% annual interest means $r = 0.03$

Example: Calculate the future value and the amount of interest if \$100 is invested for ten years at 5% compounded:

a) annually

b) monthly

Definition: **Euler's number** is written e and it has the value $e \approx 2.718$

Example: Here are the keystrokes for finding e on the Sharp and BA calculators.

Fact: Recall that m is the number of compounding periods per year. As m gets larger, the value of $(1 + \frac{r}{m})^{mt}$ gets closer to e^{rt} .

Definition: **Continuous compounding** is a theoretical scenario where interest is constantly reinvested. It roughly describes what happens when the number of compounding periods per year gets very large.

Fact: The formula for a one-time investment undergoing continuous compounding is $A = Pe^{rt}$.

Example: Calculate the future value and the amount of interest if \$100 is invested for two years at 18% compounded:

a) continuously

b) daily

Example: How much should you invest today at 6% in order to have \$10,000 in five years if interest is compounded:

a) quarterly?

b) continuously?

Definition: The **annual percentage yield**, written APY, is the rate that is actually paid. It's sometimes called the effective rate.

Fact: The APY for an investment earning compound interest is $APY = (1 + \frac{r}{m})^m - 1$.
The APY for an investment undergoing continuous compounding is $APY = e^r - 1$.

Example: Calculate the annual percentage yield for:

a) 4.95% compounded quarterly

b) 3.8% compounded continuously

8.3 Future Value of an Annuity

Definition: An **annuity** is a sequence of equal, periodic payments that earn compound interest. One example is a set of regular monthly savings towards retirement.

Fact: The formula for the future value of an annuity is

$$FV = PMT \cdot \frac{((1+i)^n - 1)}{i}, \text{ where}$$

FV is the future value, in dollars

PMT is the periodic payment, in dollars

$i = \frac{r}{m}$ is the interest rate per compounding period

$n = mt$ is the total number of compounding periods

Comment: Recall that:

r is the annual nominal interest rate, expressed as a decimal

m is the number of compounding periods per year

t is the time, in years

Comment: We are making two assumptions about all the annuities in this course. The payment period will always be the same as the compounding period. Payments will always be at the end of each compounding period.

Example: We make semi-annual deposits of \$1500 for 30 years. Our account pays 5%, compounded semi-annually. What is the value of the investment after 30 years? How much is interest?

Definition: A **sinking fund** is an investment aiming to save a specified amount of money.

Example: Quick Algebra Review

Solve for x in the equation $9 = x \cdot \frac{3}{4}$

Example: A company needs to replace a \$600,000 piece of equipment in seven years. They make quarterly payments into an account paying 9%, compounded quarterly. How much should each payment be?

8.4 Present Value of an Annuity

Definition: An **annuity** is a sequence of equal, periodic payments that earn compound interest. One example is the set of repayments on a student loan.

Fact: The formula for the present value of an annuity is

$$PV = PMT \cdot \frac{(1-(1+i)^{-n})}{i}, \text{ where}$$

PV is the present value, in dollars

PMT is the periodic payment, in dollars

$i = \frac{r}{m}$ is the interest rate per compounding period

$n = mt$ is the total number of compounding periods

Comment: Recall that:

r is the annual nominal interest rate, expressed as a decimal

m is the number of compounding periods per year

t is the time, in years

Comment: We are making two assumptions about all the annuities in this course. The payment period will always be the same as the compounding period. Payments will always be at the end of each compounding period.

Example: Your account pays 5% interest, compounded semi-annually. How much should you deposit today in order to withdraw \$2,000 every six months for the next four years? How much interest do you earn over the next four years?

Example: Quick Algebra Review

Solve for x in the equation $4 = x \cdot \frac{2}{3}$

Example: You buy a \$5,000 car and pay it off over two years at 12% interest, compounded semi-annually. What is your semi-annual payment amount?

Chapter 9: Logic

9.1 Intro to Logic

We have two main goals in this chapter:

- 1) Break down compound statements into simple statements.
- 2) Decide whether a compound statement is true or false.

These skills are important when reading legal documents such as employment contracts or tenancy agreements.

Definition: A **simple statement** is a sentence that is either true or false.

Example: Here are some simple statements. State the truth value for each one.

a) Water freezes at 23°C.

b) -2 is less than 3.

Example: The following are **not** simple statements:

a) Tomatoes are delicious.

b) $2x + 1 \geq 7$

c) Is it raining?

d) Be quiet!

Definition: Let p and q be simple statements. Here are four commonly-used **compound statements**. Let's write the notation for each compound statement.

a) p and q

b) p or q

c) not p

d) if p then q

Comment: The compound statement " p or q " means: p or q or both.

Comment: The compound statement "if p then q " can be rephrased as " p implies q ".

Example: Let p be: Paula is going to the party.

Let q be: Quinn is going to the party.

Write the following statements symbolically:

a) Quinn is not going to the party.

b) It is not the case that Paula is not going to the party.

c) Quinn and Paula are both going to the party.

d) Quinn or Paula is going to the party.

e) Paula is going to the party but Quinn is not.

f) Paula and Quinn are not both going to the party.

- g) Neither Paula nor Quinn is going to the party.
- h) Paula or Quinn is going to the party but not both.
- i) If Paula is going to the party then Quinn is going.
- j) If Quinn is going to the party then Paula is not going.

Definition: Two compound statements are **logically equivalent** if they have the same meaning.

Fact: Recall De Morgan's Laws from Section 3.2:

$$(S \cup T)' = S' \cap T'$$

$$(S \cap T)' = S' \cup T'$$

Here is the logic version of De Morgan's Laws:

$\sim (p \vee q)$ is logically equivalent to $\sim p \wedge \sim q$

$\sim (p \wedge q)$ is logically equivalent to $\sim p \vee \sim q$

Fact: Order of Operations

In a compound statement we apply $\sim$ first, then $\wedge$, then $\vee$, then $\rightarrow$, unless brackets indicate otherwise.

Example: Bracket the statements below:

a) $\sim p \wedge q$

b) $p \vee q \wedge r$

c) $q \vee \sim p \rightarrow r$

9.2 Truth Tables

In this section we'll look at how the truth values of $\sim p$, $p \vee q$, $p \wedge q$ depend on the truth values of p and q .

Fact: $\sim p$ is true exactly when p is false.

Let's build what is called the **truth table** for $\sim p$:

Fact: $p \vee q$ is true exactly when at least one of p or q is true.

Example: Build the truth table for $p \vee q$.

Fact: $p \wedge q$ is true exactly when both p and q are true.

Example: Build the truth table for $p \wedge q$.

Definition: $p \oplus q$ means: p or q , but not both.
It is pronounced “p exclusive or q” or “p xor q”.

Fact: $p \oplus q$ is true exactly when p and q have different truth values.

Example: Build the truth table for $p \oplus q$.

Example: Build the truth table for $\sim (p \wedge q)$.

Example: Build the truth table for $(p \oplus q) \vee (p \wedge q)$.

Example: Build the truth table for $(p \wedge q) \vee r$.

Example: Build the truth table for $(p \vee q) \oplus ((p \vee r) \wedge \sim p)$.

Definition: A statement that is always true is called a **tautology**. A statement that is always false is called a **contradiction**.

Example: Is the following statement a tautology, a contradiction, or neither?

$$\sim (p \vee q) \wedge q$$

9.3 If-Then Statements; If and Only If Statements

Recall that the statement “if p then q ” is written: $p \rightarrow q$.

We can think of $p \rightarrow q$ as a promise.

Example: Consider the statement “If the client likes you then you get a promotion.”
Let’s fill in the table below.

Client Likes You	You Get a Promotion	Promise is Kept

Fact: The statement $p \rightarrow q$ is false exactly when p is true and q is false.

Example: Build the truth table for $p \rightarrow q$.

Example: Build the truth table for $r \rightarrow (p \wedge q)$.

Notation: The statement “ p if and only if q ” is written: $p \leftrightarrow q$.

Fact: The statement “ p if and only if q ” is true exactly when p and q have the same truth values.

Example: Build the truth table for $p \leftrightarrow q$.

Example: Build the truth table for $(p \vee q) \leftrightarrow r$.

Fact: Order of Operations

In a compound statement we apply $\sim$ first, then $\wedge$, then $\vee$, then $\rightarrow$, then $\leftrightarrow$, unless brackets indicate otherwise.

Example: Bracket the statements below:

a) $q \wedge r \rightarrow p$

b) $\sim q \rightarrow r$

c) $p \leftrightarrow q \vee r$

Comment: There is no consensus on where $\oplus$ (exclusive or) fits into the order of operations, so statements involving $\oplus$ should always be bracketed.