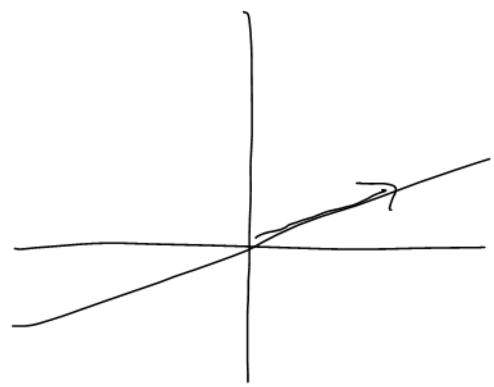
Recap: Terminology

Ex: a)



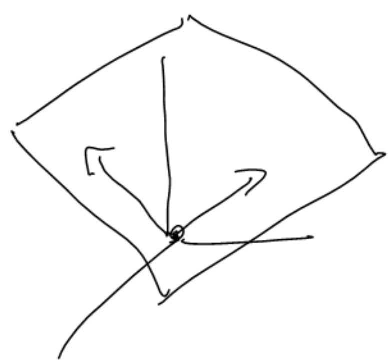
$\vec{d} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Vector form of the line is: $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} t$

A basis for the line is: $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

The line is span$\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$

b)



$\vec{d}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

$\vec{d}_2 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

Vector form of the plane is: $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} s + \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} t$

A basis for the plane is: $\left\{ \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right\}$

The plane is span$\left(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right)$

c) $\vec{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}a + \begin{bmatrix}2\\0\\1\end{bmatrix}b + \begin{bmatrix}4\\0\\2\end{bmatrix}c$

Plane through origin

Basis for the plane:

$\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix} \right\}$

or $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}4\\0\\2\end{bmatrix} \right\}$

Plane is $\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right)$

or $\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}4\\0\\2\end{bmatrix}\right)$

or $\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}4\\0\\2\end{bmatrix}\right)$

Fact: If A is upper triangular, lower triangular or diagonal then $\det A$ is the product of the diagonal entries. For example $\det \begin{bmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = -8$.

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 9 & 3 & 0 \\ 9 & 9 & 4 \end{bmatrix} = 24$$

Example: Let's understand why by calculating $\det \begin{bmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 42$$

$$= 4 \begin{vmatrix} 2 & 9 \\ 0 & -1 \end{vmatrix}$$

$$= 4 [2(-1)]$$

$$= -8$$

Fact: How Row Operations Change the Determinant:

$R_i \pm kR_j$ does not change the determinant.

$R_i \leftrightarrow R_j$ changes the sign of the determinant.

We can factor any row, for example $\det \begin{bmatrix} 3 & 6 \\ 1 & 5 \end{bmatrix} = 3 \det \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$.

Example: Calculate the determinant by reducing the matrix to REF.

Let $A = \begin{bmatrix} 1 & -2 & 1 & 9 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 4 & 5 \\ 0 & 1 & 1 & 6 \end{bmatrix}$.

$$\boxed{\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}}$$

$$|A| = \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 5 & 1 & -15 \\ 0 & 7 & 1 & -22 \\ 0 & 1 & 1 & 6 \end{vmatrix}$$

$$\boxed{R_2 \leftrightarrow R_4}$$

$$= - \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 7 & 1 & -22 \\ 0 & 5 & 1 & -15 \end{vmatrix}$$

$$\boxed{\begin{array}{l} R_3 \to R_3 - 7R_2 \\ R_4 \to R_4 - 5R_2 \end{array}}$$

$$= - \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -6 & -64 \\ 0 & 0 & -4 & -45 \end{vmatrix}$$

$$\boxed{\frac{R_3}{-6}}$$

$$= -(-6) \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{64}{6} \\ 0 & 0 & -4 & -45 \end{vmatrix}$$

$$\boxed{R_4 \to R_4 + 4R_3}$$

$$= 6 \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{64}{6} \\ 0 & 0 & 0 & -\frac{7}{3} \end{vmatrix}$$

$$= 6\left[1 \cdot 1 \cdot 1 \cdot -\frac{7}{3}\right]$$

$$= -14$$

Comment: In general $\det A \neq \det(\text{REF of } A)$.

Fact: An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Fact: Properties of $\det A$:

1) $\det A^{-1} = \frac{1}{\det A}$ (if $\det A \neq 0$)

2) $\det AB = \det A \cdot \det B$

3) $\det kA = k^n \det A$ (where A is $n \times n$) !

4) $\det A^T = \det A$

Comment: To illustrate Property 3:
$\det \begin{bmatrix} 7a & 7b \\ 7c & 7d \end{bmatrix} = 7^2 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\det \begin{bmatrix} 5a & 5b & 5c \\ 5d & 5e & 5f \\ 5g & 5h & 5i \end{bmatrix} = 5^3 \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Comment: Note that $\det(A + B) \neq \det A + \det B$ in general. !

Example: Let $\det A \neq 0$. Prove Property 1.

$\det A \neq 0$

$\Rightarrow A^{-1}$ exists

$\Rightarrow A A^{-1} = I$

$\Rightarrow \det(A A^{-1}) = \det I$

$\Rightarrow \det A \cdot \det A^{-1} = 1$

$\Rightarrow \det A^{-1} = \frac{1}{\det A}$

Fact: Cramer's Rule
Let A be an $n \times n$ matrix. When $\det A \neq 0$, the system $A\vec{x} = \vec{b}$ has a unique solution:
i-th variable= $\frac{|A_i|}{|A|}$
where $A_i = A$ with the i-th column replaced by $\vec{b}$.

Example: Solve using Cramer's Rule:

$$\begin{aligned} 2x + 3y + 2z &= -11 \\ 3x + 5z &= 23 \\ 4x + y + z &= 1 \end{aligned}$$

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 0 & 5 \\ 4 & 1 & 1 \end{vmatrix} = -3\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 47$$

$$A_1 = \begin{bmatrix} -11 & 3 & 2 \\ 23 & 0 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A_1| = \begin{vmatrix} -11 & 3 & 2 \\ 23 & 0 & 5 \\ 1 & 1 & 1 \end{vmatrix} = -3\begin{vmatrix} 23 & 5 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} -11 & 2 \\ 23 & 5 \end{vmatrix}$$

$$= -3(18) - (-101)$$

$$= 47$$

$$x = \frac{|A_1|}{|A|} = 1$$

y and z on Monday