Example: Find a basis for E_0 given $A = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

$\lambda = 0$

$[A - \lambda I \mid \vec{0}]$

$[A \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 4 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{ RREF}$$

$x_2 = t$

$x_1 + \frac{1}{4} x_2 = 0 \implies x_1 = -\frac{1}{4} t$

$x_3 = 0$

eigenvectors $\vec{x} = \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} t \quad (t \neq 0)$ or $\vec{x} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} t \quad (t \neq 0)$

Basis for $E_0 = \left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \right\}$

Fact: Let B be an $n \times n$ matrix. The system $B\vec{x} = \vec{0}$ has nontrivial solutions exactly when $\det B = 0$. (This follows from the Fundamental Theorem of Invertible Matrices). Section 3.5

Fact: To find all the eigenvalues of A: Solve the equation $\det(A - \lambda I) = 0$.

Example: Let's understand why solving $\det(A - \lambda I) = 0$ gives the eigenvalues.

λ is an eigenvalue of A

$\iff A\vec{x} = \lambda\vec{x}$ for $\vec{x} \neq \vec{0}$

$\iff (A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x} \neq \vec{0}$

$\iff \det(A - \lambda I) = 0$

Example: Find all the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$.

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & -2 \\ 5 & -7-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-7-\lambda) + 10 = 0$$

$$-28 - 4\lambda + 7\lambda + \lambda^2 + 10 = 0$$

$$\lambda^2 + 3\lambda - 18 = 0$$

$$(\lambda + 6)(\lambda - 3) = 0$$

$\lambda = -6, 3$

Example: Find a basis for E_{-6} given $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$.

$\lambda = -6$

$[A - \lambda I \mid \vec{0}]$

$[A + 6I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 10 & -2 & 0 \\ 5 & -1 & 0 \end{array}\right]$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_2 = t$

$x_1 - \frac{1}{5}x_2 = 0 \implies x_1 = \frac{1}{5}t$

eigenvectors $\vec{x} = \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} t \quad (t \neq 0)$

Basis for $E_{-6} = \left\{ \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$

Comment:

To find eigenvalues: Solve the equation $\det(A - \lambda I) = 0$.

To find eigenvectors: Solve the system $[A - \lambda I \mid \vec{0}]$. Remember to exclude $\vec{x} = \vec{0}$.

Example: Let $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. Find the eigenvectors and eigenvalues geometrically.

In a problem like this, the eigenvectors will be parallel to the x-axis and the y-axis.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -2\vec{x}$$

$\lambda = -2$

$$\boxed{\begin{array}{l} \text{eigenvectors } \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \quad (t \neq 0) \\ \text{OR Basis for } E_{-2} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \\ \text{OR } \quad E_{-2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \end{array}}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad A\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{x}$$

$\lambda = 1 \quad E_1 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

4.2 Determinants

Comment: Recall that the determinant of A is written $\det A$ or $|A|$. It's only defined for square matrices.

Fact: The cofactor expansion from Section 1.4 generalizes according to the following rules:

We can expand along any row or column.

The sign associated with each term follows the checkerboard pattern: $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$.

Example: Find $\det A$ by cofactor expansion along the second column. Calculate it again by cofactor expansion along the third row. Let $A = \begin{bmatrix} 4 & 1 & 6 \\ 1 & 2 & 3 \\ 6 & 0 & 7 \end{bmatrix}$.

2nd Column:

$|A| = -1 \left| \quad \right| + 2 \left| \quad \right| - 0 \left| \quad \right|$

$= -1 \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} + 2 \begin{vmatrix} 4 & 6 \\ 6 & 7 \end{vmatrix}$

$= -1(-11) + 2(-8)$

$= -5$

3rd row: $|A| = 6 \begin{vmatrix} 1 & 6 \\ 2 & 3 \end{vmatrix} - 0 \left| \quad \right| + 7 \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}$

$= 6(-9) + 7(7)$

$= -5$

Example: Calculate $|A|$ for $A = \begin{bmatrix} 1 & 6 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 2 & 1 & 1 & 6 \\ 2 & 0 & 5 & 7 \end{bmatrix}$.

$$\begin{bmatrix} + & & & \\ - & + & - & + \\ & & & \end{bmatrix}$$

$$|A| = 4\begin{vmatrix} 1 & 6 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 5 \end{vmatrix}$$

$$= 4\left[2\begin{vmatrix} 6 & 2 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} & \\ & \end{vmatrix} + 5\begin{vmatrix} 1 & 6 \\ 2 & 1 \end{vmatrix}\right]$$

$$= 4\left[2(4) + 5(-11)\right]$$

$$= 4(-47)$$

$$= -188$$

Example: In this example we'll illustrate the **Quick Method** for 3×3 Determinants. Calculate $\det A$ using the Quick Method. Let $A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & -2 & 6 \\ 1 & 0 & 4 \end{bmatrix}$.

$$\begin{array}{ccccc} 1 & 4 & 9 & 1 & 4 \\ 2 & -2 & 6 & 2 & -2 \\ 1 & 0 & 4 & 1 & 0 \end{array}$$

$$18 \quad 0 \quad -32 \quad -8 \quad 24 \quad 0$$

$$|A| = 18 + 0 - 32 - 8 + 24 + 0 = 2$$

Comment: The Quick Method only applies for 3×3 matrices.