Definition: The **standard matrix for** T is the matrix that performs T. It's written $[T]$.

Fact: Let $T : \mathbb{R}^2 \to \mathbb{R}^m$ be a linear transformation. To calculate $[T]$:

Place $T(\begin{bmatrix}1\\0\end{bmatrix})$ in the first column and place $T(\begin{bmatrix}0\\1\end{bmatrix})$ in the second column.

In other words, $[T] = \left[T(\begin{bmatrix}1\\0\end{bmatrix}) \;\middle|\; T(\begin{bmatrix}0\\1\end{bmatrix})\right]$.

Fact: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then:

$$[T] = \left[T(\begin{bmatrix}1\\0\\0\\\cdots\\0\end{bmatrix}) \;\middle|\; T(\begin{bmatrix}0\\1\\0\\\cdots\\0\end{bmatrix}) \;\middle|\; \cdots \;\middle|\; T(\begin{bmatrix}0\\0\\0\\\cdots\\1\end{bmatrix}) \right].$$

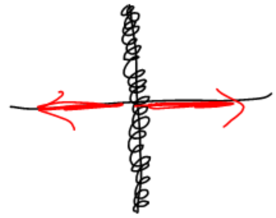
$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
and
$T(c\vec{u}) = cT(\vec{u})$

Comment: The formula for $[T]$ works because T is linear.

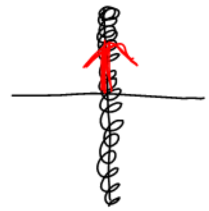
$$\begin{aligned} &T(\begin{bmatrix}x\\y\end{bmatrix}) \\ &= T(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}) \\ &= T(x\begin{bmatrix}1\\0\end{bmatrix}) + T(y\begin{bmatrix}0\\1\end{bmatrix}) \\ &= xT(\begin{bmatrix}1\\0\end{bmatrix}) + yT(\begin{bmatrix}0\\1\end{bmatrix}) \\ &= \underbrace{\begin{bmatrix}T(\begin{bmatrix}1\\0\end{bmatrix}) & T(\begin{bmatrix}0\\1\end{bmatrix})\end{bmatrix}}_{\text{standard matrix for } T} \begin{bmatrix}x\\y\end{bmatrix} \end{aligned}$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector in the y-axis. Find:

a) $[T]$



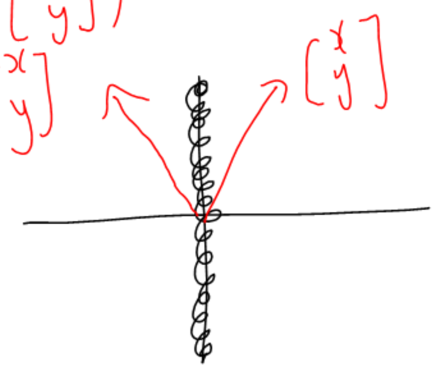
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$$



$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$[T] = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}-x\\y\end{bmatrix}$$



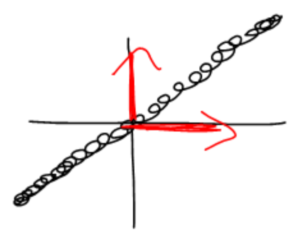
b) $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

$$= [T]\begin{bmatrix}x\\y\end{bmatrix}$$

$$= \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$= \begin{bmatrix}-x\\y\end{bmatrix}$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector in the line $y = x$. Find $[T]$.



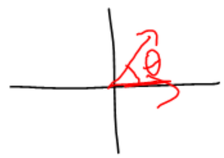
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

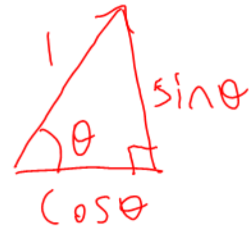
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

$$[T] = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

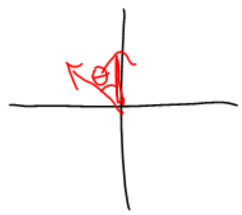
Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a vector by angle θ (counterclockwise). Find $[T]$.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$





$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$



$$[T] = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

⊗ Know this

Example: Rotate $\begin{bmatrix}1\\1\end{bmatrix}$ by $30°$ clockwise.

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}_{\theta=-30°} \begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

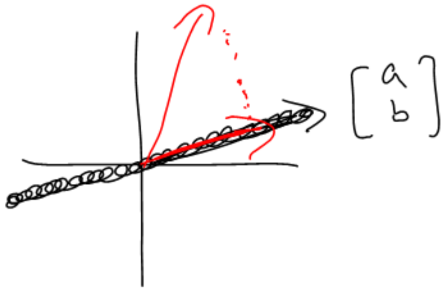
$$= \frac{1}{2}\begin{bmatrix}\sqrt{3}+1\\ -1+\sqrt{3}\end{bmatrix}$$

S	A
T	C

$\cos(-30°) = +\frac{\sqrt{3}}{2}$

$\sin(-30°) = -\frac{1}{2}$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that projects a vector on the line l through the origin with direction vector $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$. Find $[T]$.



$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \operatorname{proj}_{\begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \frac{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2} \begin{bmatrix} a \\ b \end{bmatrix}$$
$$= \frac{a}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix}$$
$$= \frac{1}{a^2+b^2} \begin{bmatrix} a^2 \\ ab \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \operatorname{proj}_{\begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{b}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix}$$
$$= \frac{1}{a^2+b^2} \begin{bmatrix} ab \\ b^2 \end{bmatrix}$$

$$[T] = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$ ⊛ Know this

Comment: It's recommended that you know the following two standard matrices:

Rotation by angle θ (counterclockwise): $[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Projection onto the line $\vec{x} = t \begin{bmatrix} a \\ b \end{bmatrix}$: $[T] = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$