Definition: Given a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$, the **coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$** is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} \text{ where } \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n.$$

Could think of it as the coefficient vector

Example: Find $[\vec{v}]_{\mathcal{B}}$ for $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \}$ and $\vec{v} = \begin{bmatrix} 5 \\ 15 \\ 28 \end{bmatrix}$.

Let $\vec{v} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 5 & 1 & 15 \\ 3 & 6 & 4 & 28 \end{array}\right]$$

$\vdots$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \text{RREF}$$

$c_1 = -2$
$c_2 = 3$
$c_3 = 4$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

Definition: The **dimension** of a subspace S is the number of vectors in a basis for S. It's written dim(S).

Comment: a) The standard basis for $\mathbb{R}^3 = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. Therefore dim $\mathbb{R}^3 = 3$.

b) dim $\mathbb{R}^n = n$
c) dim(plane through the origin in $\mathbb{R}^n$)= 2
d) dim(line through the origin in $\mathbb{R}^n$)=1

Definition: The **rank** of a matrix is the number of nonzero rows in its REF or RREF.

Comment: For any matrix A: rank(A)=dim(row(A))=dim(col(A)).

Definition: The **nullity** of a matrix A is the number of parameters in the solution to $A\vec{x} = \vec{0}$. In other words, nullity(A)=dim(null(A)).

Example: Let $A = \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Find rank$(A)$ and nullity(A).

$\xrightarrow{R_3 - R_2}$ $\begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ REF

rank$(A) = 3$

nullity$(A) = 1$

Fact: For any matrix A: rank(A)+nullity(A)= number of columns in A.

Example: Let's phrase this fact in terms of the columns of A.

$\left(\begin{array}{l}\text{\# columns} \\ \text{with pivots}\end{array}\right) + \left(\begin{array}{l}\text{\# columns} \\ \text{without pivots}\end{array}\right) = \text{\# columns of } A$

In Section 2.2 we said:
If a system is consistent then
(rank of A) + (# of parameters in solution) = # of variables.

The Fundamental Theorem of Invertible Matrices
Let A be an $n \times n$ matrix. The following statements are equivalent:

3.3
a) A is invertible.
b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d) The RREF of A is I.
e) A is a product of elementary matrices.

3.5
f) $\text{rank}(A) = n$.
g) $\text{nullity}(A) = 0$.
h) The columns of A are linearly independent.
i) The span of the columns of A is $\mathbb{R}^n$.
j) The columns of A form a basis for $\mathbb{R}^n$.
k) The rows of A are linearly independent.
l) The span of the rows of A is $\mathbb{R}^n$.
m) The rows of A form a basis for $\mathbb{R}^n$.

Ch4
n) $\det A \neq 0$.
o) 0 is not an eigenvalue of A.

Comment: Consider the Fundamental Theorem of Invertible Matrices. For a given $n \times n$ matrix, the fifteen statements are **all true** or **all false**.

Example: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 1 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 6 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$

$$= 1(14) - 2(-2) + 3(-4)$$

$$= 6$$

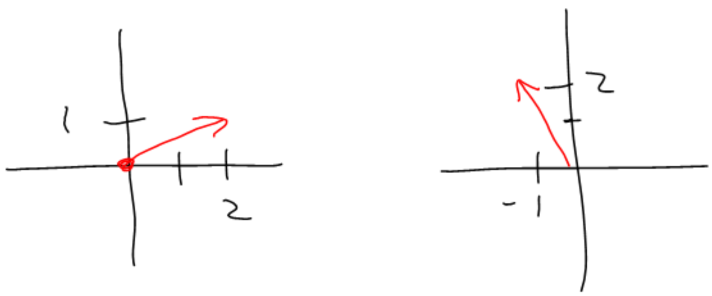
$|A| \neq 0$

Yes

3.6 Linear Transformations

Definition: A **transformation** is an operation that turns a vector into another vector.

Example: The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates a vector by $90°$ counterclockwise. Graph the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ before and after the transformation.



Definition: The vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is called the **image of** $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **under** T .

We can write $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ or $T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Definition: The **matrix transformation** T_A multiplies a vector on the left by matrix A. In other words, $T_A(\vec{x}) = A\vec{x}$.

Example: a) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -3 \end{bmatrix}$. Find $T_A(\begin{bmatrix} x \\ y \\ z \end{bmatrix})$.

$$= A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x + z \\ -x + y - 3z \end{bmatrix}$$

b) Find A given $T_A(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x + y \\ x - y \\ 3x + 3y \end{bmatrix}$.

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x - y \\ 3x + 3y \end{bmatrix}$$

Coefficients

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 3 \end{bmatrix}$$

Definition: A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if:
$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ and
$T(c\vec{u}) = cT(\vec{u})$ for all real numbers c and all vectors $\vec{u}$ in $\mathbb{R}^n$.

2 nice properties

$T(\vec{x}) = A\vec{x}$

Fact: The transformation T is linear if and only if T is a matrix transformation.

Example: Show that T is linear given $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ x \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

T is a matrix transformation
$\Rightarrow$ T is a linear transformation

Example: Show that T is not linear given $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ 1 + x \end{bmatrix}$.

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 + x \end{bmatrix}$$

No matrix M exists so that
$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 + x \end{bmatrix}$$

Note: M cannot have variables in it.

$\Rightarrow$ T is not a matrix transformation
$\Rightarrow$ T is not a linear transformation.

Recap of 3.5

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

Find a basis for:

a) row(A)

b) col(A)

c) null(A)

a)

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \text{ REF}$$

Basis for row(A) $= \{[1\ 1\ 0\ 1], [0\ 1\ -1\ 1], [0\ 0\ 0\ -2]\}$

b)

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & -2 \end{bmatrix} \text{ REF}$$

Use Columns 1, 2, 4 of A

Basis for Col(A) $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

c) $\text{null}(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \}$

$$[\, A \mid \vec{0} \,]$$

$$\left[\text{REF} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]$$

$$\begin{array}{cccc|c} x_1 & & & x_4 & \\ \hline 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \quad \text{RREF}$$

$\uparrow$

$x_3 = t$

$x_1 + x_3 = 0 \Rightarrow \quad x_1 = -t$

$x_2 = t$

$x_4 = 0$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} t$$

Basis for $\text{null}(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$