Test Review

1. Are the vectors linearly dependent? If so, write one as a linear combination of the others.

$$\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}4\\11\\1\end{bmatrix}$$

Let $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -1 & 3 & 11 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

R_2+R_1, R_3-R_1

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 5 & 15 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

$\frac{R_2}{5}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

R_1-2R_2, R_3+R_2

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

$c_3 = t$

Vectors are linearly dependent.

$c_1 - 2c_3 = 0 \Rightarrow c_1 = 2t$

$c_2 + 3c_3 = 0 \Rightarrow c_2 = -3t$

Let $t=1$: $c_1 = 2 \quad c_2 = -3 \quad c_3 = 1$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$2\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}$

$\vec{v}_3 = -2\vec{v}_1 + 3\vec{v}_2$ ✓

$\begin{bmatrix} 4 \\ 11 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ ✓

(2) Set up a system but do not solve.
Find $y = ax^3 + bx^2 + cx + d$ that passes through $(-1, 16)$, $(0, 7)$, $(1, 2)$ and $(10, 2117)$

a, b, c, d are the variables.

$$y = ax^3 + bx^2 + cx + d$$

Sub $x = -1, y = 16$:
$x = 0, y = 7$:

$$\begin{cases} 16 = -a + b - c + d \\ 7 = d \\ 2 = a + b + c + d \\ 2117 = 1000a + 100b + 10c + d \end{cases}$$

(3) Write A and A^{-1} as a product of elementary matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$R_2 - 3R_1$ $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $(R_2 + 3R_1)$

$\frac{R_2}{-2}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ $(-2R_2)$

$R_1 - 2R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ $E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $(R_1 + 2R_2)$

$$\underbrace{E_3 E_2 E_1}_{A^{-1}} A = I$$

$$A^{-1} = E_3 E_2 E_1$$

$$A = (A^{-1})^{-1}$$
$$= (E_3 E_2 E_1)^{-1}$$
$$= E_1^{-1} E_2^{-1} E_3^{-1}$$

(4) Solve using A^{-1}

$$\begin{cases} 2x - 3y = a \\ -3x + 5y = b \end{cases}$$

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$|A| = 1$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\vec{x} = A^{-1} \vec{b}$$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 5a+3b \\ 3a+2b \end{bmatrix}$$

(5) Solve for X

$$AXB = (BA)^2$$

Left multiply by A^{-1}

$$\underbrace{A^{-1}A}_{I}XB = A^{-1}(BA)^2$$

$$XB = A^{-1}(BA)^2$$

Right multiply by B^{-1}

$$X\underbrace{BB^{-1}}_{I} = A^{-1}(BA)^2 B^{-1}$$

$X = A^{-1}(BA)^2 B^{-1}$ ✓

or $X = A^{-1} BABA\, B^{-1}$ ✓

6. Find the general form of

$$\text{span}\left(\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} \right)$$

let $c_1 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Get conditions on a, b, c

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 3 & 6 & 9 & b \\ 6 & 4 & 5 & c \end{array}\right] \end{array}$$

$$\begin{array}{l} R_2 - 3R_1 \\ R_3 - 6R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 0 & 0 & b-3a \\ 0 & -8 & -13 & c-6a \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -8 & -13 & c-6a \\ 0 & 0 & 0 & b-3a \end{array}\right] \text{ REF}$$

Solvable system $\Longrightarrow$ $b - 3a = 0$

$b = 3a$

$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ such that } b = 3a \right\}$ ✓

$= \left\{ \begin{bmatrix} a \\ 3a \\ c \end{bmatrix} \right\}$ ✓