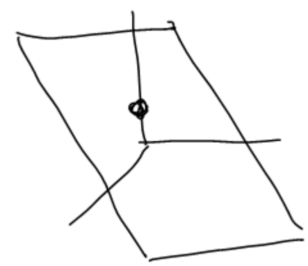
Example: Is the following set of vectors a subspace of $\mathbb{R}^3$?

$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \ z = x + 1 \}$

span(?)

$$= \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \right\}$$

$$\neq \left\{ x \begin{bmatrix} \# \\ \# \\ \# \end{bmatrix} + y \begin{bmatrix} \# \\ \# \\ \# \end{bmatrix} \right\}$$

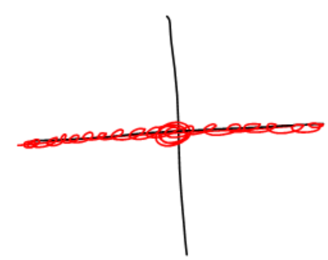


NO

Example: Is the following set of vectors a subspace of $\mathbb{R}^2$?

$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \ y = 0 \}$

$$= \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \right\}$$



$$= \left\{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$= \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

YES

Let's define three subspaces associated with a matrix A.

Definition: The **rowspace of** A is the span of the rows of A, written row(A).
The **columnspace of** A is the span of the columns of A, written col(A).
The **nullspace of** A is $\{\vec{x} \mid A\vec{x} = \vec{0}\}$, written null($A$).

Example: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

a) Is $\begin{bmatrix} 6 \\ 10 \end{bmatrix}$ in col(A)?

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 1 & 2 & 1 & 10 \end{array}\right] \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

REF

Consistent (solvable)

YES

b) Is $[1, 2, 5]$ in row(A)?

$$c_1 [1 \ 2 \ 0] + c_2 [1 \ 2 \ 1] = [1 \ 2 \ 5]$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 5 \end{array}\right] \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

REF

Consistent (solvable)

YES

c) Is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ in null(A)?

$A\begin{bmatrix}1\\2\\3\end{bmatrix} = \vec{0}$?

$\begin{bmatrix}1&2&0\\1&2&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} \neq \vec{0}$ NO

Definition: A set of vectors $\mathcal{B}$ is a **basis for a subspace** S if: span$(\mathcal{B})=S$ and $\mathcal{B}$ is linearly independent.

line through origin or plane through origin

Comment: Let's rephrase this. A set $\mathcal{B}$ is a basis for a subspace S if: $\mathcal{B}$ is a set of direction vectors for S containing the minimum number of vectors.

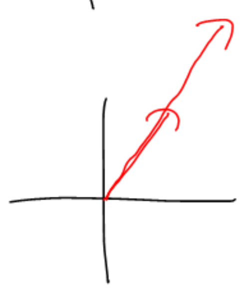
Comment: a) $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

b) $\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

c) $\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

span$\left(\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\right) \neq \mathbb{R}^2$

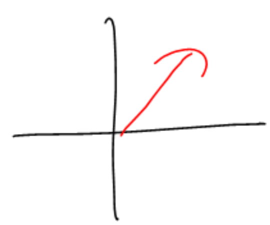
vectors are parallel



d) $\{\begin{bmatrix}3\\4\end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

span$\left(\begin{bmatrix}3\\4\end{bmatrix}\right) \neq \mathbb{R}^2$

not enough vectors



e) $\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

vectors are linearly dependent

too many vectors

Example: Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 7 & 10 \\ 8 & 17 & 8 \end{bmatrix}$. Find a basis for:

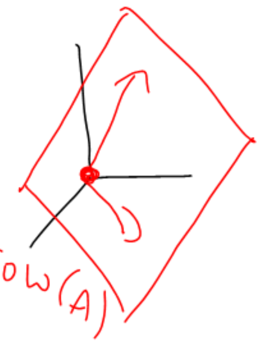
a) row(A)

Use nonzero rows of REF/RREF

$$\begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & -4 \\ 0 & 5 & -20 \end{bmatrix}$$

$$R_3 - 5R_2 \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \text{ REF}$$

Basis for $\text{row}(A) = \{ [2\ 3\ 7], [0\ 1\ -4] \}$



b) col(A)

Use columns of A corresponding to the pivots in the REF/RREF of A.

$$\text{REF of } A = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 2 of A

Basis for $\text{col}(A) = \left\{ \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 17 \end{bmatrix} \right\}$

Comment: In general, performing a row operation changes the columnspace of a matrix. We cannot use the nonzero columns of the REF/RREF to form a basis for col(A).