Test 2

Fri Mar 1

2.3 - 2.4, 3.1 - 3.3

3.4 LU Factorization

Definition: An **upper triangular matrix** is a square matrix with zeros below the main diagonal. An example is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Definition: A **lower triangular matrix** is a square matrix with zeros above the main diagonal. An example is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$.

Definition: A **unit lower triangular matrix** is lower triangular and has ones on the main diagonal. An example is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$.

Definition: The LU Factorization of a square matrix A is $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix.

Comment: Here is an LU Factorization:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

$A = LU$

upper triangular $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$

unit lower triangular $\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$

Example: Solve the system below using the LU Factorization on the previous page.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$$

System $\quad A\vec{x} = \vec{b}$

$$\underbrace{LU\vec{x}}_{\vec{y}} = \vec{b}$$

① Solve $L\vec{y} = \vec{b}$ for $\vec{y}$

② Solve $U\vec{x} = \vec{y}$ for $\vec{x}$

① Solve $L\vec{y} = \vec{b}$

$$\begin{array}{ccc} y_1 & y_2 & y_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 4 & 3 & 1 & -8 \end{array}\right]$$

$y_1 = 1$

$2y_1 + y_2 = 2 \Rightarrow 2 + y_2 = 2 \Rightarrow y_2 = 0$

$4y_1 + 3y_2 + y_3 = -8 \Rightarrow 4 + y_3 = -8 \Rightarrow y_3 = -12$

② Solve $U\vec{x} = \vec{y}$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & -12 \end{array}\right]$$

$x_1 = 1$

$x_2 = 1$

$x_3 = -2$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Fact: To find the LU Factorization of a matrix A:
Transform A to REF using only: (current row)-k(pivot row).
The matrix L has the k-values in the appropriate positions.
The matrix U is the REF.

Fact: The matrix A has an LU Factorization if and only if no row swaps are required to transform A to REF.

Example: Find the LU Factorization of $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix}$

$R_2 - 2R_1$
$R_3 - 4R_1$ $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & 9 \end{bmatrix}$ $k=2$, $k=4$

$\boxed{4 + ?(2) = 0 \quad ? = -2}$

$\boxed{8 + ?(2) = 0 \quad ? = -4}$

$R_3 - 3R_2$ $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$ $k=3$
REF

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$

1's on diagonal
k-values in appropriate positions

$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$
REF

Example: Let's explore why the method to find the LU Factorization works.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrix for $R_2 - 2R_1$:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R_2 + 2R_1)$$

L <u>undoes</u> the operations that turn A into U.

L turns U into A.

$$LU = A$$

Example: Find the LU Factorization of A and use it to solve:

$$\begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

1. Find LU

2. $\cancel{A\vec{x}=\vec{b}}$ $\quad L\underbrace{U\vec{x}}_{\vec{y}}=\vec{b}$ $\quad$ Solve $L\vec{y}=\vec{b}$ for $\vec{y}$

3. Solve $U\vec{x}=\vec{y}$ for $\vec{x}$

1.
$$\begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{l} R_2-\frac{3}{2}R_1 \\ R_3+\frac{1}{2}R_1 \end{array} \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{array}{l} k=3/2 \\ k=-\frac{1}{2} \end{array}$$

$$\boxed{\begin{array}{l} 3+?(2)=0 \\ -1+?(2)=0 \end{array}}$$

$$R_3+0R_2 \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} k=0$$

U

$$L=\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

Example Continued...

(2) Solve $L\vec{y} = \vec{b}$ for $\vec{y}$

$$
\begin{array}{cccc}
y_1 & y_2 & y_3 & \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 2 \\
3/2 & 1 & 0 & 0 \\
-1/2 & 0 & 1 & -5
\end{array}\right]
$$

$y_1 = 2$

$\frac{3}{2} y_1 + y_2 = 0 \Rightarrow 3 + y_2 = 0 \Rightarrow y_2 = -3$

$-\frac{1}{2} y_1 + y_3 = -5 \Rightarrow -1 + y_3 = -5 \Rightarrow y_3 = -4$

(3) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$

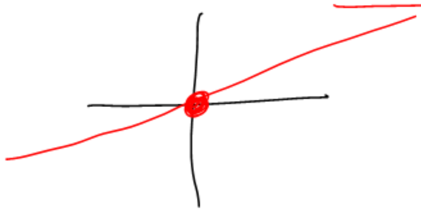
$$
\begin{array}{cccc}
x_1 & x_2 & x_3 & \\
\end{array}
\left[\begin{array}{ccc|c}
2 & -4 & 0 & 2 \\
0 & 5 & 4 & -3 \\
0 & 0 & 2 & -4
\end{array}\right]
\begin{array}{l}
x_1 = 3 \\
x_2 = 1 \\
x_3 = -2
\end{array}
$$

3.5 Subspaces and Basis

Set of linear combinations

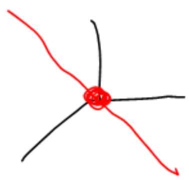
Definition: A **subspace of** $\mathbb{R}^n$ is the span of one or more vectors in $\mathbb{R}^n$.

Comment: a) A line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.



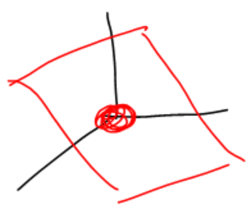
$$\vec{x} = t\vec{d}$$
$$= \text{span}(\vec{d})$$

b) A line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

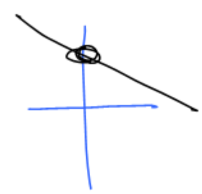


$$\vec{x} = t\vec{d}$$
$$= \text{span}(\vec{d})$$

c) A plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.



$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s\vec{u} + t\vec{v}$$
$$= s\vec{u} + t\vec{v}$$
$$= \text{span}(\vec{u}, \vec{v})$$



$$\vec{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + t\vec{d}$$
$$\vec{x} \neq \text{span}(\ )$$

This line is not a subspace of $\mathbb{R}^2$

Example: Is the following set of vectors a subspace of $\mathbb{R}^3$?

$$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 3x + 4y + z = 0 \}$$

$$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid z = -3x - 4y \}$$

$$= \{ \begin{bmatrix} x \\ y \\ -3x - 4y \end{bmatrix} \}$$

$$= \{ x \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \}$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \right)$$

$\boxed{\text{Yes}}$