Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find A^{-2}.

$$(A^2)^{-1} = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}^{-1}$$

$$= \frac{1}{4} \begin{bmatrix} 22 & -10 \\ -15 & 7 \end{bmatrix}$$

Example: Let A, B and X all be invertible $n \times n$ matrices. Solve for X given $(AX)^{-1} = BA$.

$$((AX)^{-1})^{-1} = (BA)^{-1}$$

$$AX = (BA)^{-1}$$

<u>Left</u> multiply by A^{-1}

$$\underbrace{A^{-1}A}_{I}X = A^{-1}(BA)^{-1}$$

$$X = A^{-1}(BA)^{-1}$$

or

$$X = A^{-1}A^{-1}B^{-1}$$

$$X = A^{-2}B^{-1}$$

$$X = (BA^2)^{-1}$$

Definition: An **elementary matrix** represents a row operation.
To identify which operation, consider how I has been transformed. For example:

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ represents $2R_1$

$E = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ represents -$4R_2$

$E = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ represents $R_2 + 3R_1$

$E = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$ represents $R_1 - 5R_2$

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents $R_1 \leftrightarrow R_2$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ represents $R_2 \leftrightarrow R_3$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ represents $R_2 + 6R_3$

Example: State the row operation that is represented by the elementary matrix. Then find the inverse matrix.

a) $E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

represents $3R_1$

$\frac{1}{3}R_1$ undoes it

$E_1^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

b) $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

represents $R_1 \Leftrightarrow R_2$

$R_1 \Leftrightarrow R_2$ undoes it

$E_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) $E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

represents $R_2 + 2R_1$

$R_2 - 2R_1$ undoes it

$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Fact: An elementary matrix acts on the **left** of a matrix. When an elementary matrix is multiplied on the left of A, it performs the associated row operation on A. For example: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Write A and A^{-1} as a product of elementary matrices.

$\frac{R_1}{2}$ $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ $E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $(2R_1)$

$R_1 - \frac{1}{2}R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $E_2^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $(R_1 + \frac{1}{2}R_2)$

$$\underbrace{E_2 E_1}_{A^{-1}} A = I$$

$$A^{-1} = E_2 E_1$$

$$A = (A^{-1})^{-1}$$
$$= (E_2 E_1)^{-1}$$
$$= E_1^{-1} E_2^{-1}$$

Example: Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$. Write A and A^{-1} as a product of elementary matrices.

$\frac{R_1}{2}$ $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ $E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $(2R_1)$

$R_2 - R_1$ $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $(R_2 + R_1)$

$\frac{R_2}{-1}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $(-R_2)$

$R_1 - 2R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ $E_4^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $(R_1 + 2R_2)$

$$\underbrace{E_4 E_3 E_2 E_1}_{A^{-1}} A = I$$

$$A^{-1} = E_4 E_3 E_2 E_1$$

$$A = (A^{-1})^{-1}$$
$$= (E_4 E_3 E_2 E_1)^{-1}$$
$$= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

The Fundamental Theorem of Invertible Matrices
Let A be an $n \times n$ matrix. The following statements are equivalent:
a) A is invertible.
b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d) The RREF of A is I.
e) A is a product of elementary matrices.

Comment: Consider the Fundamental Theorem of Invertible Matrices. For a given $n \times n$ matrix, the five statements are **all true** or **all false**.

Example: Consider the Fundamental Theorem of Invertible Matrices. Which of the five statements are true for A?

a) $A = \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$

$\det A \neq 0$

$\Rightarrow$ A is invertible

All 5 statements are true for A.

b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$\det A = 0$

$\Rightarrow$ A is not invertible

None of the statements are true for A.

$\left[\begin{array}{cc|c} 1 & 4 & \# \\ 6 & 9 & \# \end{array}\right]$ has 1 solution.

$\left[\begin{array}{cc|c} 1 & 2 & \# \\ 3 & 6 & \# \end{array}\right]$ has infinitely-many solutions or no solution.