If ATA = I then A is <u>invertible</u> and A⁻¹ is the inverse of A "A inverse" **Definition:** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the **determinant of** A is det A = ad - bc.

Fact: If A is a 2×2 matrix then:

$$A^{-1} = \begin{cases} \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, & \text{if } \det A \neq 0 \\ \text{undefined}, & \text{if } \det A = 0 \end{cases}$$

Example: Find A^{-1} :

a)
$$A = \begin{bmatrix} 1 & -4 \\ 7 & 2 \end{bmatrix}$$

 $de = A = 2 - (-28) = 30$
 $A^{-1} = \frac{1}{30} \begin{bmatrix} 2 & 4 \\ -7 & 1 \end{bmatrix}$

b)
$$A = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$$

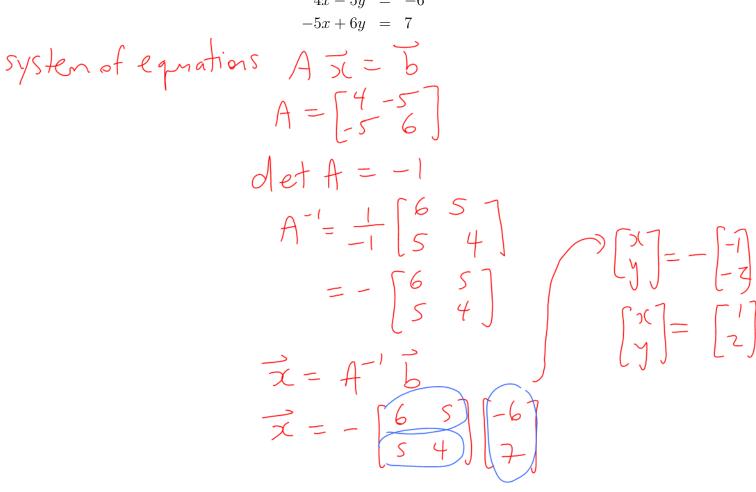
 $det A = 18 - 18 = 0$
 A^{-1} does not exist
 $\begin{pmatrix} A & is not invertible \end{pmatrix}$

Fact: If A^{-1} exists then the system of equations $A\vec{x} = \vec{b}$ has a unique solution: $\vec{x} = A^{-1}\vec{b}$. **Example:** Let's explore why the above fact is true. 5

System of equations
$$A\vec{x} = \vec{b}$$

Cetticents Tables Gestarts
 $\vec{A} \cdot \vec{A} \cdot \vec{x} = \vec{A} \cdot \vec{b}$
 $\vec{x} = \vec{A} \cdot \vec{b}$





Fact: To find A^{-1} for an $n \times n$ matrix we form the augmented matrix [A|I]. We perform row operations to produce I on the left side. The resulting matrix on the right side will be A^{-1} .

Example: Find
$$A^{-1}$$
 for $A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

$$\begin{bmatrix} A & | I] \\ 2 & 5 & | & | & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 2 & | & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 5 & 1 & | & 0 & 0 \\ 0 & 2 & 2 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & | & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & | & 0 & 0 & 1 \\ R_{2} - 2R_{1} & \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & -3 & | & 1 & -2 & 0 \\ 0 & 1 & -3 & | & 1 & -2 & 0 \\ 0 & -2 & -2 & | & 0 & -2 & 1 \end{bmatrix}$$

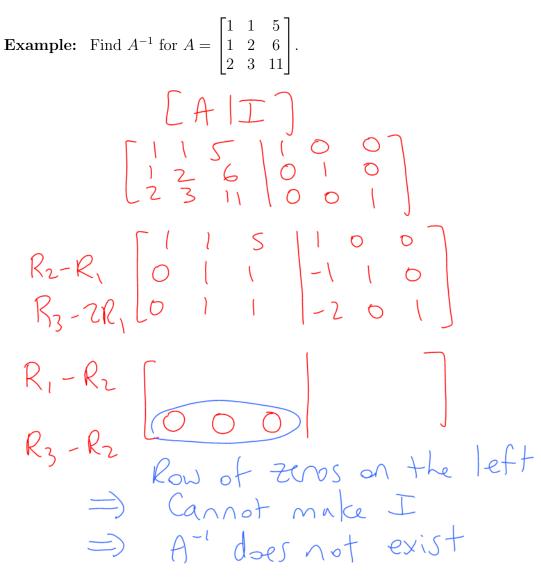
$$R_{1} - 2R_{1} & \begin{bmatrix} 1 & 0 & 8 & | & -2 & 5 & 0 \\ 0 & 1 & -3 & | & 1 & -2 & 0 \\ 0 & 0 & -8 & | & 2 & -6 & 1 \end{bmatrix}$$

$$R_{3} + 2R_{2} & \begin{bmatrix} 1 & 0 & 8 & | & -2 & 5 & 0 \\ 0 & 1 & -3 & | & 1 & -2 & 0 \\ 0 & 0 & -8 & | & 2 & -6 & 1 \end{bmatrix}$$

$$R_{1} - 8R_{3} & \begin{bmatrix} 1 & 0 & 0 & | & 0 & -1 & 1 \\ 0 & 1 & 0 & | & \frac{2}{8} & \frac{2}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & | & \frac{2}{8} & \frac{2}{8} & -\frac{3}{8} \end{bmatrix}$$

$$R_{1} - 8R_{3} & \begin{bmatrix} 1 & 0 & 0 & | & 0 & -1 & 1 \\ 0 & 1 & 0 & | & \frac{2}{8} & \frac{2}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & | & \frac{2}{8} & \frac{2}{8} & -\frac{3}{8} \end{bmatrix}$$

Comment: By transforming A into I we are "undoing" A. The matrix on the right side will be the matrix that "undoes" A, that is A^{-1} .



Fact: Suppose a zero row appears on the left side while reducing [A|I]. Then A^{-1} does not exist.

We'll look at three properties of A^{-1} .

Property 1: If A^{-1} exists then $(A^{-1})^{-1} = A$.

Property 2: $(A^T)^{-1} = (A^{-1})^T$ for any matrix *A*.

Example: Verify Property 2 for $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. $\begin{pmatrix} A^{\top} \end{pmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$ $\begin{pmatrix} A^{-1} \end{pmatrix}^{\top} = \begin{pmatrix} 1 & (7 & -2) \\ -3 & 1 \end{pmatrix}^{\top} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}^{\top} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$

Property 3: For any matrices A_1, A_2, \ldots, A_n with compatible sizes: $(A_1A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$.

Comment: In particular this means that $(AB)^{-1} = B^{-1}A^{-1}$.

Comment: Let Operation A represent putting on your socks. Let Operation B represent putting on your shoes. To reverse this sequence we have to undo the operations and **reverse** the order of operations. We could express this in matrix terms as $(AB)^{-1} = B^{-1}A^{-1}$.

Comment: Consider Property 3 with all *n* matrices equal to *A*. The statement becomes $(A^n)^{-1} = (A^{-1})^n$. This means we can write A^{-n} without confusion.