Test 2

Fri Mar 1

2.3, 2.4, 3.1–3.3

6 Questions

Bring calculator

Bring music/earplugs

Practice Problems on website

Definition: Matrices A and B **commute** if $AB = BA$.

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix}$. Do A and B commute?

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 11 & 39 \\ 13 & 37 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 39 \\ 13 & 37 \end{bmatrix}$$

Yes

We're going to look at six properties of matrices.

Property 1: For any matrices A, B and C with compatible sizes:
$(AB)C = A(BC)$

Example: Verify Property 1 for $A = \begin{bmatrix} 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 6 \end{bmatrix}$.

$$(AB)C = \left(\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right) \begin{bmatrix} 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -10 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -60 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 3 \end{bmatrix} \left(\begin{bmatrix} 2 \\ -4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 12 \\ -4 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -60 \end{bmatrix} \checkmark$$

Property 2: For any matrices A, B and C with compatible sizes:
$A(B+C) = AB + AC$

Example: Verify Property 2 for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 50 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix} + \begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} 22 \\ 50 \end{bmatrix}$$

Properties 3 and 4: For any matrix A:
$AI = A$ and $IA = A$ (Section 3.1)

Property 5: For any matrices A and B with compatible sizes:
$(A \pm B)^T = A^T \pm B^T$

Example: Break Property 5 into two statements.

$$(A+B)^T = A^T + B^T$$
$$(A-B)^T = A^T - B^T$$

Example: Confirm that $(A-B)^T = A^T - B^T$ for $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$.

$$(A-B)^T = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

$$A^T - B^T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

!

Property 6: For any matrices $A_1, A_2, \ldots, A_n$ with compatible sizes:
$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$

Example: Confirm that $(AB)^T = B^T A^T$ for $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$.

$$(AB)^T = \begin{bmatrix} 3 & 8 \\ 4 & 10 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 \\ 8 & 10 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 8 & 10 \end{bmatrix}$$

Why does the order reverse?

Transposing reverses rows and columns.

Example: Expand $(A+B)^2$ and simplify.

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= AA + AB + BA + BB\\ &= A^2 + AB + BA + B^2\end{aligned}$$

Example: Show that A^TA is symmetric.

M is symmetric means $M^T = M$ (Section 3.1)

A^TA " $(A^TA)^T = A^TA$

Start with the more complicated side.

$$\begin{aligned}(A^TA)^T &= A^T(A^T)^T \quad \text{(Property 6)}\\ &= A^T A \ \checkmark\end{aligned}$$

e.g.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} \cdot & * & \xi \\ * & \cdot & \Delta \\ \xi & \Delta & \cdot \end{bmatrix}$

3.3 The Inverse of a Matrix

Definition: An $n \times n$ matrix A is **invertible** if there exists a matrix A^{-1} so that $AA^{-1} = I$ and $A^{-1}A = I$.

Definition: The matrix A^{-1} is called the **inverse of A**. Informally: "A inverse"

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$. Confirm that $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Real #	Matrices
$\frac{1}{3} \cdot 3 = 1$	$A^{-1}A = I$

Comment: 1) Not every square matrix is invertible.
2) If A^{-1} exists then it is unique. (A matrix can't have two inverses.)
3) $AA^{-1} = I$ if and only if $A^{-1}A = I$, so we only need to check one property.