Example: Simplify B^{2018} given that $B^3 = I$.

$2018 = 3(?) + ?$

$\frac{2018}{3} \approx 672.7$

$2018 = 3(672) + ?$

$2018 = 3(672) + 2$

$B^{2018} = B^{3(672)+2}$

$= B^{3(672)} B^2$

$= (B^3)^{672} B^2$

$= I^{672} B^2$

$= I B^2$

$= B^2$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$IA = A$

Definition: Let **O** be the zero matrix. For example **O**$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or **O**$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

Example: Find a 2×2 matrix A so that $A^2 =$**O** but $A \neq$**O**.

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

Many possible answers.

3.2 Matrix Algebra

Example: Is $\begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix}$?

Let $c_1 \begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$

$$\begin{cases} 0c_1 + 0c_2 = 0 \\ 1c_1 + 1c_2 = 1 \\ 6c_1 + 7c_2 = 4 \end{cases}$$

$\therefore$

$$\begin{array}{cc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 6 & 7 & 4 \\ 2 & 2 & 2 \end{array}\right]$$

Reorder rows $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 6 & 7 & 4 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$

$R_2 - 6R_1$
$R_3 - 2R_1$ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ REF

Solvable system $\Longrightarrow$ YES

To check: $R_1 - R_2$ $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ RREF

$c_1 = 3$
$c_2 = -2$

$$3\begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$$

Example: Find the general form of span($\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$).

$$c_1 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Goal: Get conditions on w, x, y, z.
Each zero row of REF gives a condition.

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & w \\ 0 & 0 & 1 & x \\ 2 & 6 & 4 & y \\ 0 & 1 & 5 & z \end{array}\right]$$

$$R_3 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & w \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & y-2w \\ 0 & 1 & 5 & z \end{array}\right]$$

Reorder rows
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & w \\ 0 & 1 & 5 & z \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & y-2w \end{array}\right] \text{ REF}$$

Solvable system $\Rightarrow$ $y - 2w = 0$
$y = 2w$

$$\left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \text{ such that } y = 2w \right\}$$

$$= \left\{ \begin{bmatrix} w & x \\ 2w & z \end{bmatrix} \right\}$$

Comment: The general form of the span allows us to quickly identify which matrices are in the span. For example, $\begin{bmatrix} 1 & 7 \\ 2 & 30 \end{bmatrix}$ is in the span and $\begin{bmatrix} 1 & 7 \\ 3 & 30 \end{bmatrix}$ is not.

Example: Find the general form of span$\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}\right)$.

$$c_1 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & x \\ 2 & 4 & y \\ 0 & 5 & z \end{array}\right]$$

$$R_3 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & x \\ 0 & 0 & y-2w \\ 0 & 5 & z \end{array}\right]$$

$$R_4 - 5R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & x \\ 0 & 0 & y-2w \\ 0 & 0 & z-5x \end{array}\right] \text{ REF}$$

Solvable system

$$\Rightarrow \quad y - 2w = 0 \quad \text{and} \quad z - 5x = 0$$
$$y = 2w \qquad\qquad z = 5x$$

$$\left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \text{ such that } y = 2w \text{ and } z = 5x \right\}$$

$$= \left\{ \begin{bmatrix} w & x \\ 2w & 5x \end{bmatrix} \right\}$$

e.g. $\begin{bmatrix} 1 & 7 \\ 2 & 35 \end{bmatrix}$ is in the span

$\begin{bmatrix} 3 & 17 \\ 6 & 12 \end{bmatrix}$ is not in the span

Example: Are $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ linearly independent?

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix} + c_3\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = 0$

Yes (linearly independent)

Infinitely-many solutions c_1, c_2, c_3

No (linearly dependent)

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 6 & 4 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]} \end{array}$$

$R_3 - 2R_1$ $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$

Reorder rows $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ REF

1 solution ($c_1 = c_2 = c_3 = 0$)

Yes

Ex: Express the concept using algebra

a) a linear combination of $\vec{u}$ and $\vec{v}$

Many possible answers.

$$4\vec{u} + 7\vec{v}$$

$$8\vec{u} - 12\vec{v}$$

$$c_1\vec{u} + c_2\vec{v}$$

b) $\text{span}(\vec{u}, \vec{v})$

$$\{ c_1\vec{u} + c_2\vec{v} \}$$

c) $\vec{x}$, $\vec{y}$ and $\vec{z}$ are linearly independent

$$c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \vec{0}$$

and $c_1 = c_2 = c_3 = 0$

OR

$c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \vec{0}$

has 1 solution

d) $\vec{x}$, $\vec{y}$ and $\vec{z}$ are linearly dependent

$c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \vec{0}$

has infinitely-many solutions c_1, c_2, c_3

<u>Ex</u>: Can $c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \vec{0}$ be unsolvable?

No