Test #5

How many solutions?

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 64 & 8 \end{array}\right]$$

$$R_2 - kR_1 \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 64-k^2 & 8-k \end{array}\right]$$

$64 - k^2 \neq 0$

$$\frac{R_2}{64-k^2} \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{8-k}{64-k^2} \end{array}\right]$$

1 solution

$64 - k^2 = 0$

$(8-k)(8+k) = 0$

$k = 8$

$$\left[\begin{array}{cc|c} 1 & 8 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

∞-many solutions

$k = -8$

$$\left[\begin{array}{cc|c} 1 & -8 & 1 \\ 0 & 0 & 16 \end{array}\right]$$

no solution

Comment: $A+B$ is undefined if A and B have different sizes.

Definition: The process of multiplying a matrix by a real number is called **scalar multiplication**.

Example: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$. Find $A - 3B$.

$$A - 3B$$
$$= A + (-3B)$$
$$= \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 9 \\ -3 & -18 & -27 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 6 & 10 \\ -5 & -20 & -23 \end{bmatrix}$$

Definition: The **transpose** of A, written A^T, interchanges the rows and columns of A. The matrix A is **symmetric** if $A^T = A$.

Example: Calculate the transpose and state if the matrix is symmetric.

a) $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$$

A is symmetric

b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}$

$$B^T = \begin{bmatrix} 1 & 0 \\ 2 & 6 \\ 1 & 1 \end{bmatrix}$$

B is <u>not</u> symmetric.

Fact: To multiply two matrices:

$$AB = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & \dots & r_1 \cdot c_n \\ r_2 \cdot c_1 & r_2 \cdot c_2 & \dots & r_2 \cdot c_n \\ \dots & \dots & \dots & \dots \\ r_n \cdot c_1 & r_n \cdot c_2 & \dots & r_n \cdot c_n \end{bmatrix}$$

where r_i is row i of matrix A and c_j is column j of matrix B.

Example: Find AB where $A = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}$

$$[1\ 4] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1(1) + 4(0) = 1$$

$$AB = \begin{bmatrix} 1 & 9 & 27 \\ -2 & 0 & 0 \end{bmatrix}$$

$$[-2\ 1] \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 0$$

Example: Let's consider the sizes of A and B in the example above.
Which two numbers must be equal to make AB defined?
Which two numbers predict the size of AB?

$$\underbrace{(2 \times \underbrace{2)(2}_{\text{must be equal}} \times 3)}_{\text{size of } AB}$$

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix}$$

Example: Let A be a 2×3 matrix and let B be a 3×1 matrix. Calculate the sizes of AB and BA.

AB: $(2 \times \underbrace{3)(3}_{= \checkmark} \times 1)$ AB is 2×1

BA: $(3 \times \underbrace{1)(2}_{\neq} \times 3)$ BA is undefined

Fact: $AB \neq BA$ in general.

Example: Find BC and CB where $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

$$BC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 11 & 13 \\ 17 & 19 \end{bmatrix}$$

$$CB = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix}$$

CB is undefined

Example: Expand the following:
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{cases} x + 2y = 5 \\ 3x + 4y = 6 \end{cases}$$

Comment: Matrix multiplication was reverse-engineered to solve systems of equations.

Fact: A system of equations can be written as $A\vec{x} = \vec{b}$ where:
A is the coefficient matrix
$\vec{x}$ is the vector of variables, written as a column
$\vec{b}$ is the vector of constants, written as a column

Example: Consider the data below:

	Al	Bob
Test1 Mark	50	60
Test2 Mark	90	80
Exam Mark	75	70

Test1 Weight	Test2 Weight	Exam Weight
0.2	0.2	0.6

Let A be a matrix containing the course marks for the two students. Let B be a matrix containing the weightings of the coursework. Find Al and Bob's final grades using a matrix multiplication.

$$A = \begin{array}{c} \\ T1 \\ T2 \\ E \end{array}\begin{array}{c} \begin{array}{cc} Al & Bob \end{array} \\ \begin{bmatrix} 50 & 60 \\ & \ldots \\ & \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \begin{array}{ccc} T1 & T2 & E \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

Need compatible sizes $\underline{\text{and}}$ compatible categories.

$$BA = \begin{array}{c} \begin{array}{ccc} T1 & T2 & E \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ T1 \\ T2 \\ E \end{array}\begin{array}{c} \begin{array}{cc} Al & Bob \end{array} \\ \begin{bmatrix} 50 & 60 \\ 90 & 80 \\ 75 & 70 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} Al & Bob \end{array} \\ \begin{bmatrix} 73 & 70 \end{bmatrix} \end{array}$$

Definition: The **outer product expansion of** AB is:
$AB = A_1B_1 + A_2B_2 + \ldots + A_nB_n$
where A_i is column i of A and B_j is row j of B.

Comment: Normal matrix multiplication involves **rows** of the first matrix and **columns** of the second matrix.
The outer product expansion involves **columns** of the first matrix and **rows** of the second matrix.

Example: Find the outer product expansion of AB given:
$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 4 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 7 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}\begin{bmatrix} 4 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 6 \\ -8 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 13 & 13 \\ -8 & -4 \end{bmatrix}$$

Example: Confirm the result in the previous example using normal matrix multiplication.

$$AB = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 4 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 13 & 13 \\ -8 & -4 \end{bmatrix} \quad \checkmark$$

Comment: The outer product expansion will be used further in Section 5.4.

Definition: The expression A^n means multiply A with itself n times. For example:
$A^2 = AA$
$A^3 = A^2A$ or $A^3 = AA^2$ or $A^3 = AAA$

Example: Express A^{12} as the cube of a matrix.

$$A^{12} = (A^4)^3$$

Example: Compute A^2 for $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$$

Fact: Recall that I is the identity matrix. For any matrix A:
$AI = A$ and $IA = A$

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Confirm that $AI = A$ and $IA = A$.

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \checkmark$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \checkmark$$