Example: Balance $NH_3 + O_2 \to N_2 + H_2O$

$$w\,NH_3 + x\,O_2 \to y\,N_2 + z\,H_2O$$

w, x, y, z are the variables

$$\begin{aligned}
N:&\quad w = 2y &\implies&\quad w - 2y = 0\\
H:&\quad 3w = 2z &\implies&\quad 3w - 2z = 0\\
O:&\quad 2x = z &\implies&\quad 2x - z = 0
\end{aligned}$$

$$\begin{array}{ccccc} w & x & y & z & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0\\ 3 & 0 & 0 & -2 & 0\\ 0 & 2 & 0 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{2}{3} & 0\\ 0 & 1 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & -\frac{1}{3} & 0 \end{array}\right]$$

$z = t$

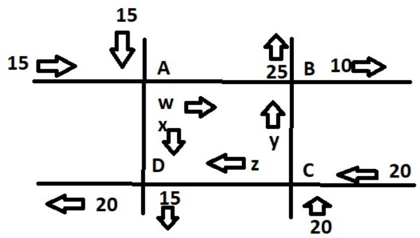
$$\begin{aligned}
w - \tfrac{2}{3}z = 0 &\implies w = \tfrac{2}{3}t\\
x - \tfrac{1}{2}z = 0 &\implies x = \tfrac{1}{2}t\\
y - \tfrac{1}{3}z = 0 &\implies y = \tfrac{1}{3}t
\end{aligned}$$

Choose smallest positive integer solution : $t = 6$

$[w, x, y, z] = [4, 3, 2, 6]$ ✓

$4\,NH_3 + 3\,O_2 \to 2N_2 + 6H_2O$ ✓

Example: Consider the following network of one-way streets. The average number of vehicles per hour through intersections A,B,C,D was collected from historical data.
a) Find the flows w, x, y, z.
b) If the solution has a parameter then specify the possible values of the parameter.



Inflow = Outflow at each intersection.

$$\begin{aligned}
A:&\quad 15+15 = w+x &&\Rightarrow\quad w+x=30\\
B:&\quad w+y = 25+10 &&\Rightarrow\quad w+y=35\\
C:&\quad 20+20 = y+z &&\Rightarrow\quad y+z=40\\
D:&\quad x+z = 20+15 &&\Rightarrow\quad x+z=35
\end{aligned}$$

$$\begin{array}{cccc} w & x & y & z & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 30\\ 1 & 0 & 1 & 0 & 35\\ 0 & 0 & 1 & 1 & 40\\ 0 & 1 & 0 & 1 & 35 \end{array}\right]$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -5\\ 0 & 1 & 0 & 1 & 35\\ 0 & 0 & 1 & 1 & 40\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t \qquad (t \geq 0)$

$$\begin{aligned}
w-z&=-5 &&\Rightarrow\quad w=-5+t &&(t\geq 5)\\
x+z&=35 &&\Rightarrow\quad x=35-t &&(t\leq 35)\\
y+z&=40 &&\Rightarrow\quad y=40-t &&(t\leq 40)
\end{aligned}$$

$(5 \leq t \leq 35)$

$$\begin{bmatrix} w\\ x\\ y\\ z \end{bmatrix} = \begin{bmatrix} -5\\ 35\\ 40\\ 0 \end{bmatrix} + t\begin{bmatrix} 1\\ -1\\ -1\\ 1 \end{bmatrix}$$

Example: Find all possible combinations of 15 coins (nickels, dimes or quarters) that total \$ 2.50.

x = # of nickels
y = " dimes
z = " quarters

$$x + y + z = 15$$

$$5x + 10y + 25z = 250 \quad (¢)$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 5 & 10 & 25 & 250 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -20 \\ 0 & 1 & 4 & 35 \end{array}\right]$$

$z = t$ $\quad (t \geq 0)$

$x - 3z = -20 \implies x = -20 + 3t \quad (t \geq 7)$

$y + 4z = 35 \implies y = 35 - 4t \quad (t \leq 8)$

Want non-negative integer solutions.

$(t = 7 \text{ or } 8)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 7 \end{bmatrix} \quad (t=7) \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} \quad (t=8)$$

Chapter 3: Matrices

3.1 Matrix Operations

Definition: The **size** of a matrix is given by (# of rows)×(# of columns).

For example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a 2×3 matrix.

Definition: The **entry** of a matrix A is written a_{ij} or $[A]_{ij}$, where i and j are the row index and the column index respectively. For the matrix above $a_{23} = 6$ or $[A]_{23} = 6$.

Definition: A **square** matrix has size $n \times n$.

Definition: An **identity** matrix is square with ones along the main diagonal and zeros elsewhere. It can be written I, or I_n if we want to emphasize its size.

For example $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Definition: A **diagonal** matrix is square and all the entries off the main diagonal are zero.

For example $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$. Find:

a) $A + B$

$$= \begin{bmatrix} 2 & 6 & -2 \\ -1 & 4 & 13 \end{bmatrix}$$

b) $3A$

$$= \begin{bmatrix} 3 & 18 & 3 \\ -6 & -6 & 12 \end{bmatrix}$$

Comment: $A + B$ is undefined if A and B have different sizes.

Definition: The process of multiplying a matrix by a real number is called **scalar multiplication**.

Example: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$. Find $A - 3B$.

Definition: The **transpose** of A, written A^T, interchanges the rows and columns of A. The matrix A is **symmetric** if $A^T = A$.

Example: Calculate the transpose and state if the matrix is symmetric.

a) $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}$