**Comment:** To decide if a system is consistent, reduce it to REF.
To solve a system, reduce it to RREF.

**Definition:** Given $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, consider solutions to $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$. If the only solution is $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly independent**. If there are solutions other than $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly dependent**.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$$

$c_1 = c_2 = \ldots = c_n = 0$ is the only solution

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly <u>independent</u>

$c_1 = c_2 = \ldots = c_n = 0$ and other solutions

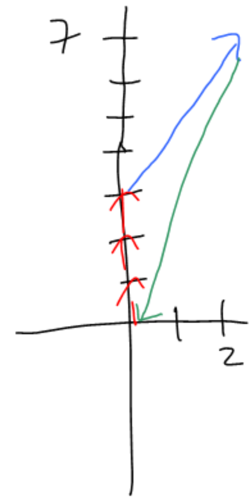
$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly <u>dependent</u>

**Comment:** The two sentences below mean the same thing:
Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.
The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent.

**Comment:** The two sentences below mean the same thing:
Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent.
The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent.

**Comment:** a) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}$ is linearly dependent.

$$3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$




b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ are linearly dependent.

$$0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Coefficients must be not all zero

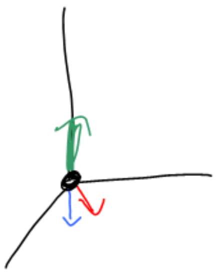
c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly dependent.

$$1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

at least one Coefficient must be nonzero

**Example:** Are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\2\end{bmatrix}$ linearly independent?

Let $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$



$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\frac{R_2}{-1} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ R_3 + R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\frac{R_3}{2} \quad \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array} \text{RREF}$$

$c_1 = 0$

$c_2 = 0$

$c_3 = 0$

$\boxed{\text{Yes}}$

**Fact:** A set of more than $n$ vectors in $\mathbb{R}^n$ is linearly dependent. For example three vectors in $\mathbb{R}^2$ are guaranteed to be linearly dependent.

**Example:** Let's explore why this fact is true.

Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \vec{0}$

$$n\left\{\underbrace{\left[\begin{array}{cccc|c} c_1 & c_2 & \cdots & c_m & 0 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{array}\right]}_{m}\right.$$

$(m > n)$

The system is solvable because $c_1 = c_2 = \dots = c_m = 0$ is a solution.

RREF: $\left[\begin{array}{ccccc|c} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \end{array}\right]$

At least 1 parameter in solution.

$\Rightarrow$ System has infinitely-many solutions $c_1, c_2, \dots, c_m$

$\Rightarrow$ Vectors are linearly dependent.

**Example:** Find a linear dependence relationship (linear dependency) involving $\begin{bmatrix}1\\6\end{bmatrix}$, $\begin{bmatrix}2\\6\end{bmatrix}$ and $\begin{bmatrix}4\\30\end{bmatrix}$. Start by letting $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 6 & 6 & 30 & 0 \end{array}\right] \end{array}$$

$$R_2 - 6R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right]$$

$$\frac{R_2}{-6} \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_1 - 2R_2 \quad \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$$c_3 = t$$

$$c_1 + 6c_3 = 0 \Rightarrow c_1 = -6t$$

$$c_2 - c_3 = 0 \Rightarrow c_2 = t$$

Choose any nonzero $t$ : $t = 1$

$$c_1 = -6 \qquad c_2 = 1 \qquad c_3 = 1$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$-6\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$$

**Example:** Find a linear dependence relationship (linear dependency) involving $\begin{bmatrix}1\\6\end{bmatrix}$, $\begin{bmatrix}2\\6\end{bmatrix}$ and $\begin{bmatrix}4\\30\end{bmatrix}$. Start by putting the vectors into the rows of a matrix.

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{array} \begin{bmatrix} 1 & 6 \\ 2 & 6 \\ 4 & 30 \end{bmatrix}$$

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 - 2\vec{v}_1 \\ \vec{v}_3 - 4\vec{v}_1 \end{array} \begin{bmatrix} 1 & 6 \\ 0 & -6 \\ 0 & 6 \end{bmatrix}$$

$$(\vec{v}_3 - 4\vec{v}_1) + (\vec{v}_2 - 2\vec{v}_1) \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Any zero row leads to a linear dependency.

$$\vec{v}_3 - 4\vec{v}_1 + \vec{v}_2 - 2\vec{v}_1 = \vec{0}$$

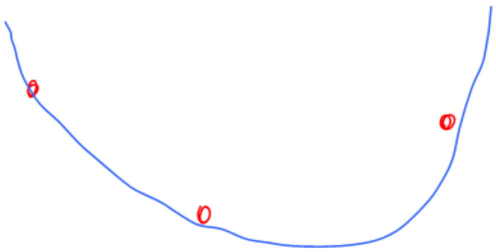
$$-6\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$$

**Comment:** Compare the methods used in the last two examples. The first method gives the general solution, while the second method gives one particular solution.

**Comment:** Preview of Section 3.5:
We'll consider objects like lines or planes through the origin, and find a set of direction vectors containing the minimum number of vectors. This discussion will require knowledge of span and linear independence.

## 2.4 Applications of Linear Systems

**Example:** Find the parabola $y = ax^2 + bx + c$ that passes through $(1, 12), (-1, 18)$ and $(2, 30)$.



$y = ax^2 + bx + c$

$a, b, c$ are the variables

$$ax^2 + bx + c = y$$

Sub $x=1, y=12$: $\quad a + b + c = 12$

$x=-1, y=18$: $\quad a - b + c = 18$

$x=2, y=30$: $\quad 4a + 2b + c = 30$

$$\begin{array}{cccc} a & b & c & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & -1 & 1 & 18 \\ 4 & 2 & 1 & 30 \end{array}\right]} \end{array}$$

$$\rightsquigarrow \begin{array}{cccc} a & b & c & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{array}\right]} \end{array} \qquad \begin{pmatrix} a = 7 \\ b = -3 \\ c = 8 \end{pmatrix}$$

$$y = ax^2 + bx + c$$

$$y = 7x^2 - 3x + 8$$

**Example:** Balance $NH_3 + O_2 \to N_2 + H_2O$

$$w\,NH_3 + x\,O_2 \to y\,N_2 + z\,H_2O$$

$w, x, y, z$ are the variables

N: $w = 2y \Rightarrow w - 2y = 0$

H: $3w = 2z \Rightarrow 3w - 2z = 0$

O: $2x = z \Rightarrow 2x - z = 0$

$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right]$$

To Be Continued