Test Review

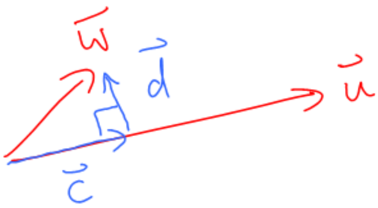
Ex: Let $\vec{u} = [1, -1, 2]$ and $\vec{w} = [3, 0, 4]$.

Find $\vec{c}$ and $\vec{d}$ so that:

$\vec{c}$ is parallel to $\vec{u}$,

$\vec{d}$ is perpendicular to $\vec{u}$, and

$\vec{w} = \vec{c} + \vec{d}$.



$$\vec{c} = \text{proj}_{\vec{u}}\, \vec{w}$$

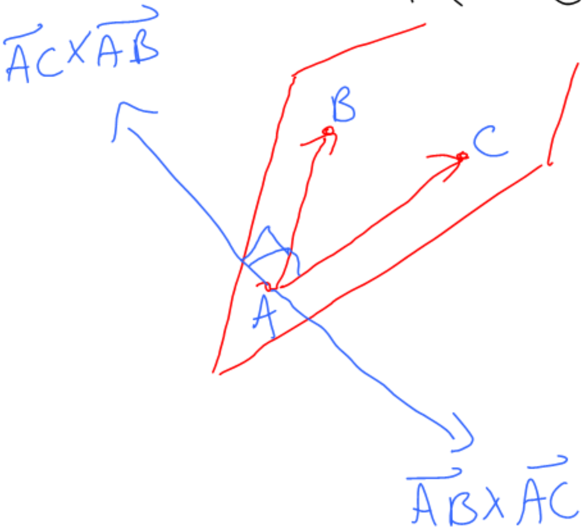
$$\vec{d} = \vec{w} - \vec{c}$$

$$\vec{c} = \text{proj}_{\vec{u}}\, \vec{w} = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\|^2}\vec{u} = \frac{11}{6}[1, -1, 2]$$

$$\vec{w} = \vec{c} + \vec{d}$$

$$\vec{d} = \vec{w} - \vec{c} = [3, 0, 4] - \frac{11}{6}[1, -1, 2] = \left[\frac{7}{6}, \frac{11}{6}, \frac{2}{6}\right] = \frac{1}{6}[7, 11, 2]$$

Ex: Normal form of the plane through $A=(1,2,3)$, $B=(6,5,4)$ and $C=(3,-3,-3)$?



$$\vec{AB} = [5,3,1]$$
$$\vec{AC} = [2,-5,-6]$$

$$\vec{AC}\times\vec{AB} = [13,-32,31]$$

$$\boxed{\begin{array}{cccccc} 2 & -5 & -6 & 2 & -5 \\ & \times & \times & \times & \\ 5 & 3 & 1 & 5 & 3 \end{array}}$$

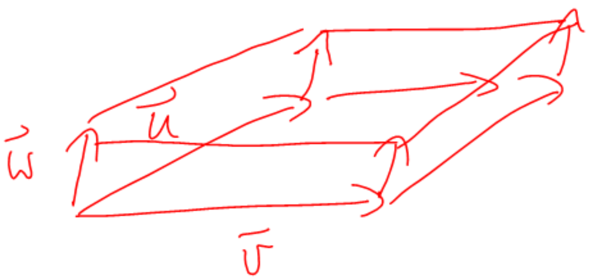
$$\vec{n} = [13,-32,31]$$

$$\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p}$$

$$\begin{bmatrix}13\\-32\\31\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}13\\-32\\31\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}$$

OR $$\begin{bmatrix}-13\\32\\-31\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-13\\32\\-31\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Ex: Volume of parallelepiped formed by $\bar{u} = [-4, 2, 3]$, $\bar{v} = [2, 1, 2]$ and $\bar{w} = [3, -3, 6]$?



slanted box

$$\begin{bmatrix} + & - & + \\ - & + & \end{bmatrix}$$

$$V = \left| \begin{vmatrix} -4 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & -3 & 6 \end{vmatrix} \right|$$

$$= \left| -4 \begin{vmatrix} 1 & 2 \\ -3 & 6 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & 6 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix} \right|$$

$$= \left| -4(12) - 2(6) + 3(-9) \right|$$

$$= |-87|$$

$$= 87$$

Ex: Solve using Gauss-Jordan Elimination:

$$\begin{cases} 2x - 4y + 6z = 12 \\ 3x + 5y + z = 7 \\ 10x + 2y + 14z = 38 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 12 \\ 3 & 5 & 1 & 7 \\ 10 & 2 & 14 & 38 \end{array}\right]$$

$$\frac{R_1}{2} \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 3 & 5 & 1 & 7 \\ 10 & 2 & 14 & 38 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 10R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 11 & -8 & -11 \\ 0 & 22 & -16 & -22 \end{array}\right]$$

$$\frac{R_2}{11} \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 1 & -\frac{8}{11} & -1 \\ 0 & 22 & -16 & -22 \end{array}\right]$$

$$\begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 22R_2 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & \frac{17}{11} & 4 \\ 0 & 1 & -\frac{8}{11} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad 3 - \frac{16}{11}$$

RREF

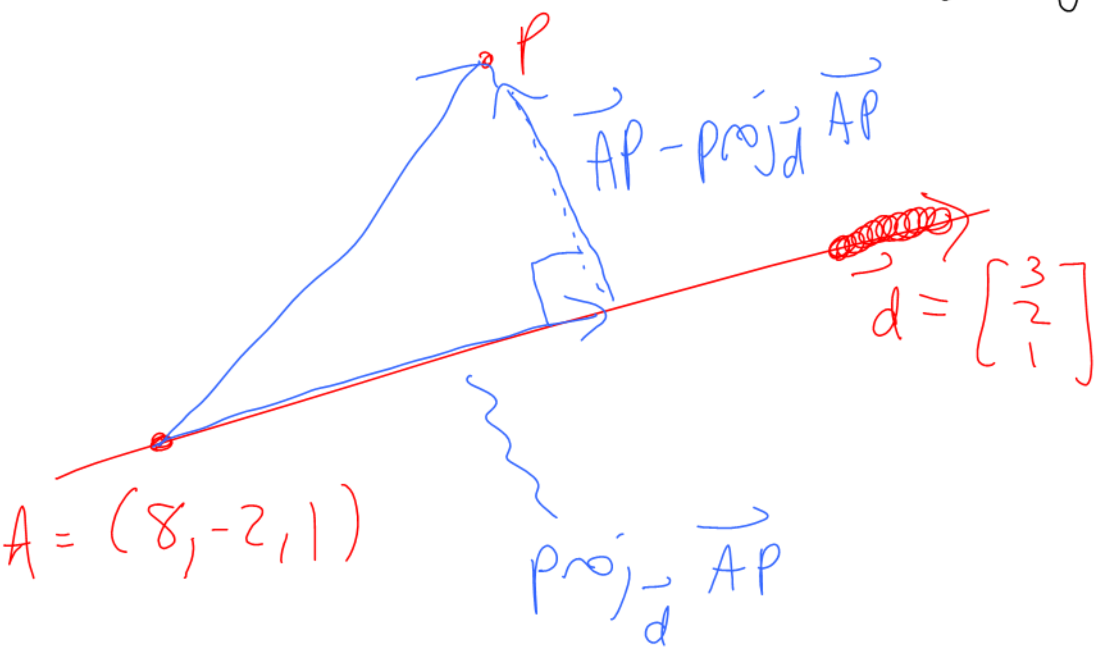
$$z = t$$

$$x + \frac{17}{11} z = 4 \Rightarrow x = 4 - \frac{17}{11} t$$

$$y - \frac{8}{11} z = -1 \Rightarrow y = -1 + \frac{8}{11} t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -17/11 \\ 8/11 \\ 1 \end{bmatrix}$$

Ex: Find the distance between $P = (6,1,2)$ and $\vec{x} = \begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.



$$\text{distance} = \| \vec{AP} - \text{proj}_{\vec{d}} \vec{AP} \|$$

$$\vec{AP} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$\text{proj}_{\vec{d}} \vec{AP} = \frac{\vec{d} \cdot \vec{AP}}{\|\vec{d}\|^2} \vec{d}$$

$$= \frac{1}{14} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{AP} - \text{proj}_{\vec{d}} \vec{AP} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} - \frac{1}{14} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -31/14 \\ 40/14 \\ 13/14 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} -31 \\ 40 \\ 13 \end{bmatrix}$$

$$\text{distance} = \left\| \frac{1}{14} \begin{bmatrix} -31 \\ 40 \\ 13 \end{bmatrix} \right\|$$

$$= \frac{1}{14} \sqrt{2730}$$

$$\approx 3.73$$

Ex: Let $\vec{u} = \begin{bmatrix} x \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Find x if:

a) $\vec{u}$ is parallel to $\vec{v}$

b) $\vec{u}$ is perpendicular to $\vec{v}$

c) $\vec{u}$ and $\vec{v}$ make an angle of $30°$

a)
$$\vec{u} = k\vec{v}$$
$$\begin{bmatrix} x \\ 2 \end{bmatrix} = k\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

2nd component: $2 = k(3)$

$k = \frac{2}{3}$

1st component: $x = k(4)$

$= \frac{8}{3}$

b)
$$\vec{u} \cdot \vec{v} = 0$$
$$4x + 6 = 0$$
$$x = -\frac{3}{2}$$

c)
$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$
$$4x + 6 = \sqrt{x^2+4}\,\sqrt{25}\left(\frac{\sqrt{3}}{2}\right)$$
$$8x + 12 = 5\sqrt{3}\sqrt{x^2+4}$$

Square both sides: $64x^2 + 192x + 144 = 25(3)(x^2+4)$

$$64x^2 + 192x + 144 = 75x^2 + 300$$

$$0 = 11x^2 - 192x + 156$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{192 \pm \sqrt{30000}}{22}$$

$$\approx 16.6,\ 0.85$$