2.3 Span and Linear Independence

Example: Is $\begin{bmatrix} 8 \\ -10 \end{bmatrix}$ a linear combination of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

Let $c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$

$$\begin{cases} -c_1 + 2c_2 = 8 \\ 2c_1 - 3c_2 = -10 \end{cases}$$

$$\begin{array}{ccc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 2 & 8 \\ 2 & -3 & -10 \end{array}\right]$$

$$\frac{R_1}{-1} \left[\begin{array}{cc|c} 1 & -2 & -8 \\ 2 & -3 & -10 \end{array}\right]$$

$$R_2 - 2R_1 \left[\begin{array}{cc|c} 1 & -2 & -8 \\ 0 & 1 & 6 \end{array}\right] \text{ REF}$$

System is Consistent. YES

To check: $R_1 + 2R_2$ $\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right] \end{array}$ RREF

$c_1 = 4$
$c_2 = 6$

$$4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix} \checkmark$$

Example: Is $\vec{w} = \begin{bmatrix}1\\1\\2\end{bmatrix}$ a linear combination of $\vec{u} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\3\\0\end{bmatrix}$?

Let $\underbrace{c_1\vec{u} + c_2\vec{v}}_{\text{linear combination of } \vec{u} \text{ and } \vec{v}} = \vec{w}$

$$\begin{array}{cc}c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{array}\right]$$

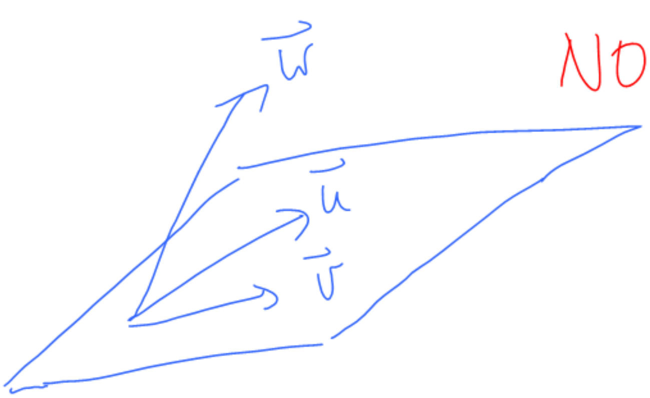
$$R_3 - R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{array}\right]$$

$$\frac{R_2}{3} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & -1 & 1 \end{array}\right]$$

$$R_3 + R_2 \quad \left[\begin{array}{cc|c} & & \\ & & \\ 0 & 0 & \frac{4}{3} \end{array}\right] \quad \text{REF}$$

No solution.

NO



Fact: The vector $\vec{b}$ is a linear combination of the columns of matrix A if and only if the system $\left[A|\ \vec{b}\right]$ is consistent.

Definition: The **span** of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is the set of all linear combinations of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

Comment: a) span($\vec{a}, \vec{b}$)= $\{\vec{0},\ 3\vec{a},\ -7\vec{b},\ 2\vec{a}+5\vec{b}, \ldots\}$

b) span($\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$)= $\{c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n\}$
where $c_1, c_2, \ldots, c_n$ are any real numbers.

Fact: The zero vector $\vec{0}$ is in span$(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ because $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_n = \vec{0}$.

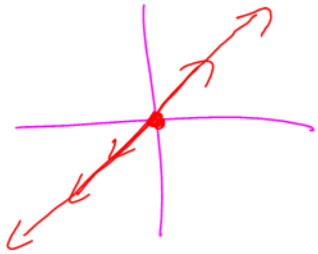
Example: Describe each span geometrically:

a) span($\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}-3\\-3\end{bmatrix}$)

$= \{ c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}-3\\-3\end{bmatrix} \}$

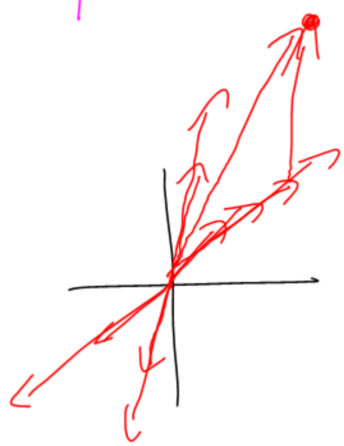
$= \{ t \begin{bmatrix}1\\1\end{bmatrix} \}$

Line through origin in $\mathbb{R}^2$ with $\vec{d} = \begin{bmatrix}1\\1\end{bmatrix}$



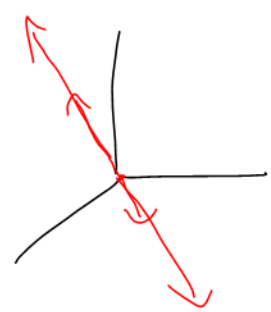
b) span($\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\end{bmatrix}$)

All of $\mathbb{R}^2$ (xy-plane)



c) span($\begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\begin{bmatrix}-4\\0\\4\end{bmatrix}$)

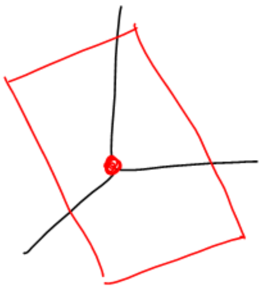
Line through origin in $\mathbb{R}^3$ with $\vec{d} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$



d) span($\begin{bmatrix}1\\6\\0\end{bmatrix}$, $\begin{bmatrix}2\\3\\0\end{bmatrix}$)

Plane through origin in $\mathbb{R}^3$ with direction vectors $\vec{u} = \begin{bmatrix}1\\6\\0\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}2\\3\\0\end{bmatrix}$.

Example: Find an equation for span$\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\6\end{bmatrix}\right)$. Give your answer in any form.



Plane through origin in $\mathbb{R}^3$.

Method I: Vector Form

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

$$\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}1\\3\\6\end{bmatrix}$$

Method II: Normal Form

$$\vec{n} = \vec{u} \times \vec{v}$$

$$= [-3, -5, 3]$$

$$\boxed{\begin{array}{ccccc} 1 & 0 & 1 & 1 & 0 \\ 1 & 3 & 6 & 1 & 3 \end{array}}$$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

$$\begin{bmatrix}-3\\-5\\3\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-3\\-5\\3\end{bmatrix} \cdot \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Example: a) Show that span($\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$)= $\mathbb{R}^2$. (algebra)

Let $c_1\begin{bmatrix}1\\3\end{bmatrix}+c_2\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$

Show this is solvable for c_1, c_2.

$$\begin{array}{cc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 3 & 1 & b \end{array}\right]$$

$$R_2-3R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -5 & b-3a \end{array}\right] \quad \text{REF}$$

System is consistent ✓

b) Write $\begin{bmatrix}a\\b\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$.

$$\frac{R_2}{-5} \quad \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & \frac{b-3a}{-5} \end{array}\right]$$

$$\boxed{\begin{aligned} &a+\tfrac{2}{5}(b-3a) \\ &= a+\tfrac{2}{5}b-\tfrac{6}{5}a \\ &= -\tfrac{1}{5}a+\tfrac{2}{5}b \end{aligned}}$$

$$R_1-2R_2 \quad \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & \frac{-a}{5}+\frac{2b}{5} \\ 0 & 1 & \frac{3a}{5}-\frac{b}{5} \end{array}\right] \end{array} \quad \text{RREF}$$

$$c_1 = -\frac{a}{5}+\frac{2b}{5}$$

$$c_2 = \frac{3a}{5}-\frac{b}{5}$$

$$\left(\frac{-a}{5}+\frac{2b}{5}\right)\begin{bmatrix}1\\3\end{bmatrix}+\left(\frac{3a}{5}-\frac{b}{5}\right)\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$