Example: Find the intersection of the two lines:

$\vec{x} = \begin{bmatrix} -5 \\ 6 \\ 5 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vdots$

$$\begin{array}{ccc} s & t & \# \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -6 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & -1 & 0 \\ -1 & -1 & -6 \end{array}\right]$

$\begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$ $\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 4 \\ 0 & -2 & -8 \end{array}\right]$

$\begin{array}{l} R_1 + R_2 \\ \\ R_3 + 2R_2 \end{array}$ $\begin{array}{ccc} s & t & \\ \end{array}$ $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right] \leftarrow$ no info

$s = 2$
$t = 4$

Sub $s=2$ into 1^{st} line or $t=4$ into 2^{nd} line

$\vec{x} = \begin{bmatrix} -1 \\ 8 \\ 3 \end{bmatrix}$

Example: How many solutions does the following system have?

$$\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right]$$

$$R_2 - kR_1 \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right]$$

$1-k^2 \neq 0$

$$\frac{R_2}{1-k^2} \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{1-k}{1-k^2} \end{array}\right]$$

1 solution

$1-k^2 = 0$

$(1-k)(1+k) = 0$

$k=1$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

∞-many solutions

$k=-1$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ \boxed{0} & \boxed{0} & \boxed{2} \end{array}\right]$$

no solution

$$\begin{cases} 1 \text{ solution}, & \text{if } 1-k^2 \neq 0 \\ \infty\text{-many}, & \text{if } k = 1 \\ \text{no solution}, & \text{if } k = -1 \end{cases}$$

Definition: The **rank** of a matrix is the number of nonzero rows in its REF or RREF.

solvable

Fact: If a system is consistent then:
rank+(# of parameters in solution)=# of variables

Example: Verify the fact for the following system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

rank = 2
# of variables = 3
# of parameters in solution = 1

Example: Rephrase the fact in terms of columns of the coefficient matrix.

(# of columns with pivots) + (# of columns without pivots) = (# of columns)

rank + (# of parameters) = # of variables

Comment: Notice that $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a solution to the following system:

$$\begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 0 \end{aligned}$$

Definition: A system whose constants are all zero is called a **homogeneous system**. The solution $\vec{x} = \vec{0}$ is called the **trivial solution**.

Fact: A homogeneous system always has at least one solution: $\vec{x} = \vec{0}$.

Example: Consider a homogeneous system with more variables than equations. How many solutions does the system have?

$$m\left\{\left[\begin{array}{c|c} & 0 \\ \underbrace{\qquad\qquad\qquad\qquad}_{n} & 0 \\ & \vdots \\ & 0 \end{array}\right]\right. \qquad (n > m)$$

At least one solution: $\vec{x} = \vec{0}$

RREF: $\left[\begin{array}{ccccc|c} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{array}\right]$

At least 1 column without a pivot

$\Rightarrow$ Infinitely-many solutions.

2.2 #45

Find the intersection of 2 planes:

$$\begin{cases} 3x + 2y + z = -1 \\ 2x - y + 4z = 5 \end{cases}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array}\right] \end{array}$$

$$\frac{R_1}{3} \quad \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 2 & -1 & 4 & 5 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{7}{3} & \frac{10}{3} & \frac{17}{3} \end{array}\right]$$

$$-1 - \frac{4}{3}$$

$$-\frac{3}{7} \times R_2 \quad \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{10}{7} & -\frac{17}{7} \end{array}\right]$$

$$\frac{1}{3} - \frac{2}{3}\left(-\frac{10}{7}\right)$$

$$R_1 - \frac{2}{3}R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{7} & \frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{17}{7} \end{array}\right] \text{ RREF}$$

$$z = t$$

$$x + \frac{9}{7}z = \frac{9}{7} \Rightarrow x = \frac{9}{7} - \frac{9}{7}t$$

$$y - \frac{10}{7}z = -\frac{17}{7} \Rightarrow y = -\frac{17}{7} + \frac{10}{7}t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9/7 \\ -17/7 \\ 0 \end{bmatrix} + \begin{bmatrix} -9/7 \\ 10/7 \\ 1 \end{bmatrix} t$$