2.1 Linear Systems

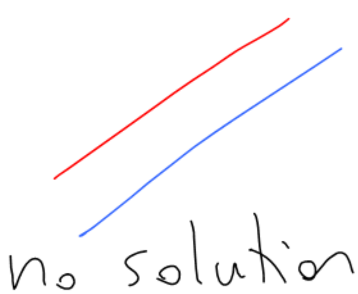
Definition: A **linear equation** in $\mathbb{R}^2$ has the form $ax + by = c$, where a, b and c are real numbers.

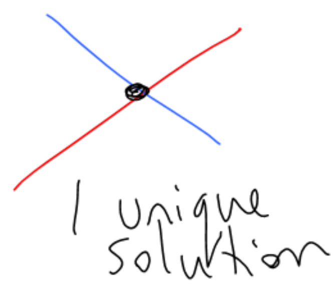
Definition: A **linear system** in $\mathbb{R}^2$ consists of two or more linear equations. It's often just called a **system**.

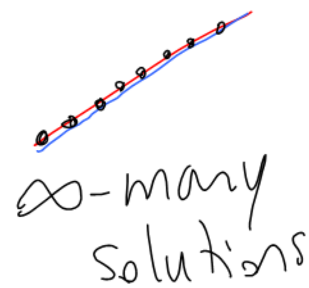
Comment: Here's an example of a system:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

Fact: A system can have: no solution, one unique solution or infinitely-many solutions.







Definition: A system with no solution is called an **inconsistent system**. (unsolvable)
A **consistent system** has one solution or infinitely-many solutions. In other words, a consistent system is solvable.

Definition: Consider the system:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

The matrix $\begin{bmatrix} 2 & 6 \\ -3 & 3 \end{bmatrix}$ is called the **coefficient matrix**.

The matrix $\left[\begin{array}{cc|c} 2 & 6 & -14 \\ -3 & 3 & -15 \end{array}\right]$ is called the **augmented matrix**.

Fact: There are three types of elementary row operations that can be performed on an augmented matrix. These row operations don't change the solution of the system:
1) Swap two rows
2) Multiply or divide a row by a nonzero real number
3) (Current Row) $\pm$ #(Pivot Row)

Example: Solve by elimination:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

$$\begin{array}{ccc} x & y & \# \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 6 & -14 \\ -3 & 3 & -15 \end{array}\right]$$

Get a 1, "the pivot"

$$\frac{R_1}{2} \quad \left[\begin{array}{cc|c} 1 & 3 & -7 \\ -3 & 3 & -15 \end{array}\right]$$

Get 0's in rest of Column 1

$$R_2 + 3R_1 \quad \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & 12 & -36 \end{array}\right]$$

Current Row − # (Pivot Row)

Get a 1, "the pivot"

$$\frac{R_2}{12} \quad \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & 1 & -3 \end{array}\right]$$

Get 0's in rest of Column 2

$$R_1 - 3R_2 \quad \begin{array}{ccc} x & y & \# \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

Current Row − # (Pivot Row)

$$1x + 0y = 2 \Rightarrow x = 2$$
$$0x + 1y = -3 \Rightarrow y = -3$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Example: Solve:

$$\begin{aligned} 2x - 3y &= 8 \\ -4x + 6y &= 20 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -4 & 6 & 20 \end{array}\right]$$

Get a 1

$$\frac{R_1}{2} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ -4 & 6 & 20 \end{array}\right]$$

Get 0's in Column 1

$$R_2 + 4R_1 \left[\begin{array}{cc|c} & & \\ 0 & 0 & 36 \end{array}\right]$$

Current Row − # (Pivot Row)

$0x + 0y = 36$ impossible

System has no solution.

Fact: A system has no solution if the following type of row appears while performing row operations:
[all zeros | nonzero]

Example: Solve:

$$\begin{aligned} 2x - 3y &= 8 \\ -4x + 6y &= -16 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -4 & 6 & -16 \end{array}\right]$$

$$\frac{R_1}{2} \quad \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ -4 & 6 & -16 \end{array}\right]$$

$$\begin{array}{c} \\ R_2+4R_1 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & \# \end{array} \\ \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 0 \end{array}\right] \end{array} \quad 0x+0y=0$$

No INFO

Column for y has no pivot.

y is a <u>free variable</u>

$\boxed{y=t}$ (t is a <u>parameter</u>)

$1x - \frac{3}{2}y = 4 \Rightarrow x = 4 + \frac{3}{2}y \Rightarrow \boxed{x = 4 + \frac{3}{2}t}$

$$(y = 0 + 1t) \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

System has infinitely-many solutions.

Example: Solve:

$$\begin{aligned} x &= 5 \\ 2x + 3y &= 4 \\ 3x + 4y &= 7 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{array}\right]$$

Get 0's in rest of Column 1

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 3 & -6 \\ 0 & 4 & -8 \end{array}\right]$$

$$\frac{R_2}{3} \quad \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{array}\right]$$

$$R_3 - 4R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{no info}$$

$$x = 5$$
$$y = -2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

1 unique solution

Definition: **Back-substitution** is the process of solving a sytem from the bottom equation upwards.

Example: Solve by back-substitution:

$$\begin{aligned} 4x + \ y + \ \ z &= 15 \\ 3y + 5z &= 29 \\ 2z &= 8 \end{aligned}$$

$2z = 8 \Rightarrow z = 4$

$3y + 5z = 29 \Rightarrow 3y + 20 = 29 \Rightarrow y = 3$

$4x + y + z = 15 \Rightarrow 4x + 3 + 4 = 15 \Rightarrow x = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Comment: Most systems can't be solved by back-substitution.

2.2 Solving Systems

Definition: A matrix is in **row-echelon form** (REF) if:
any zero rows are at the bottom AND
the leading nonzero entries of each row move down and right

Comment: The following matrices are in REF:
$$\begin{bmatrix} 6 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition: An augmented matrix is in REF if the coefficient matrix is in REF.

Comment: The following matrices are in REF:
$$\left[\begin{array}{ccc|c} 6 & 0 & -1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

Definition: One method of solving a system is **Gaussian Elimination**. The augmented matrix is transformed to REF using elementary row operations. The system is then solved by back-substitution.

Example: Solve by Gaussian Elimination:

$$\begin{aligned} x + 2y + z &= 6 \\ 2x + 2y &= 8 \\ 3y + z &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 2 & 0 & 8 \\ 0 & 3 & 1 & 8 \end{array}\right]$$

$$R_2 - 2R_1 \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -2 & -2 & -4 \\ 0 & 3 & 1 & 8 \end{array}\right]$$

$$\frac{R_2}{-2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 1 & 8 \end{array}\right]$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ R_3 - 3R_2 & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 2 \end{array}\right] \end{array} \quad \text{REF}$$

Back-substitute

$$-2z = 2 \implies z = -1$$

$$y + z = 2 \implies y - 1 = 2 \implies y = 3$$

$$x + 2y + z = 6 \implies x + 6 - 1 = 6 \implies x = 1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Definition: A matrix is in **reduced row-echelon form** (RREF) if:
the matrix is in REF,
the leading nonzero entry in each row is 1, AND
these leading ones have zeros everywhere else in their columns

Comment: The following matrices are in RREF:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comment: The following matrix is in REF but not RREF:

$$\begin{bmatrix} 1 & \underline{2} & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition: An augmented matrix is in RREF if the coefficient matrix is in RREF.

Comment: The following matrices are in RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 5 & 0 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

Definition: Another method of solving a system is **Gauss-Jordan Elimination**. The augmented matrix is transformed to RREF using elementary row operations. This is typically faster than Gaussian Elimination.