(30) Gram-Schmidt

Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$4\vec{v}_2 = 4\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-2)}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$6\vec{v}_3 = 6\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ -2 \\ -2 \end{bmatrix}$$

Orthogonal Basis $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -2 \\ -2 \end{bmatrix} \right\}$

(31) $A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T}$

$\vec{q}_1, \vec{q}_2$: orthonormal eigenvectors, written as columns.

$\lambda_1 = 2 \qquad \vec{q}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$

$\lambda_2 = -2 \qquad \vec{q}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix}-3\\1\end{bmatrix}$

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T}$$

$$= 2\frac{1}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1 & 3\end{bmatrix} - 2\frac{1}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{bmatrix}-3\\1\end{bmatrix}\begin{bmatrix}-3 & 1\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix} - \frac{1}{5}\begin{bmatrix}9 & -3\\-3 & 1\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}-8 & 6\\6 & 8\end{bmatrix}$$

33 Let $y = a_0 + a_1 x$

a_0, a_1 are the variables

$$a_0 + a_1 x = y$$

$$1(a_0) + x(a_1) = y$$

$$\begin{array}{cc} 1 & x \end{array} \qquad\qquad\qquad y$$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 2\\1 & 3\\1 & 4\end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix} = \begin{bmatrix}2\\3\\5\\7\end{bmatrix}$$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

$$= \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}}$$

$$= \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 17 \\ 51 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 0 \\ 34 \end{bmatrix} \begin{matrix} \leftarrow a_0 = 0 \\ \leftarrow a_1 = \frac{34}{20} = 1.7 \end{matrix}$$

$$y = a_0 + a_1 x$$

$$\boxed{y = 1.7x}$$

35 $|z| = \sqrt{5^2 + 12^2} = 13$

$$\theta = \tan^{-1}\left(\frac{12}{5}\right) \quad (+\pi?)$$

$$= \tan^{-1}\frac{12}{5}$$

$$z = |z|\left[\cos\theta + i\sin\theta\right]$$

$$z = 13\left[\cos\left(\tan^{-1}\frac{12}{5}\right) + i\sin\left(\tan^{-1}\frac{12}{5}\right)\right]$$

Follow-Up:

Find z^4

$$z^4 = 13^4\left[\cos\left(4\tan^{-1}\frac{12}{5}\right) + i\sin\left(4\tan^{-1}\frac{12}{5}\right)\right]$$