Suggested HW for Complex Numbers
Omit #7

Assignment due Monday

Example: Calculate i^0, i^1, i^2, i^3, i^4 and i^5.

$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i$

$i^4 = 1 \quad i^5 = i \quad \ldots$

Fact: Let n be a non-negative integer. Then:
$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1$ and $i^{4n+3} = -i$.

Example: Simplify i^{271}.

$\frac{271}{4} = 67 + 0.75$

$271 = 4(67) + 3$

$$i^{271} = i^{4(67)+3} = i^{4(67)} \cdot i^3 = (i^4)^{67} \cdot i^3 = 1\,(-i) = -i$$

Example: Recall that:
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Show that $e^{i\theta} = \cos\theta + i\sin\theta$.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \ldots$$

$$= \left[1 - \frac{\theta^2}{2!} + \ldots\right] + i\left[\theta - \frac{\theta^3}{3!} + \ldots\right]$$

$$= \cos\theta + i\sin\theta$$

Example: Derive the **most beautiful equation in mathematics** by subbing $\theta = \pi$ into the equation $e^{i\theta} = \cos\theta + i\sin\theta$.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\theta = \pi$: $$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

Combines the 5 most important constants:

$0, 1, \pi, e, i$

Definition: The **rectangular form** of a complex number is $z = a + bi$.
The **polar form** of a complex number is $z = |z|[\cos\theta + i\sin\theta]$.
The **exponential form** of a complex number is $z = |z|e^{i\theta}$.

$\boxed{\text{Phasor Notation} \quad z = |z| \angle \theta}$

Now we'll look at complex eigenvalues and eigenvectors.

Example: Let $A = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}$.

a) Find the eigenvalues.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(1-\lambda) + 65 = 0$$

$$\lambda^2 - 4\lambda + 3 + 65 = 0$$

$$\lambda^2 - 4\lambda + 68 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(68)}}{2}$$

$$= \frac{4 \pm \cancel{\sqrt{-256}}}{2} \quad \cancel{\sqrt{256}}\sqrt{-1} \quad 16i$$

$$= 2 \pm 8i$$

b) Find a basis for one of the eigenspaces.

$\lambda = 2+8i$:

$$\left[A - (2+8i) I \mid \vec{0} \right]$$

$$\left[\begin{array}{cc|c} 1-8i & -13 & 0 \\ 5 & -1-8i & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 5 & -1-8i & 0 \\ 1-8i & -13 & 0 \end{array}\right]$$

$$\frac{R_1}{5} \quad \left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 1-8i & -13 & 0 \end{array}\right]$$

Method #1

$$\left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

because the system has nontrivial solutions.

Method #2

$$R_2 - (1-8i)R_1 \quad \left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 0 & & 0 \end{array}\right]$$

$$-13 - \frac{(1-8i)(-1-8i)}{5}$$

$$= -13 + \frac{(1-8i)(1+8i)}{5}$$

~~c) Find a basis for the other eigenspace.~~

$$= -13 + \frac{65}{5}$$
$$= 0$$

$$\text{RREF} = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$x_2 = t$$

$$x_1 + \left(\frac{-1-8i}{5}\right)x_2 = 0 \implies x_1 = \frac{1+8i}{5}t$$

$$\vec{x} = \begin{bmatrix} \frac{1+8i}{5} \\ 1 \end{bmatrix} t \qquad \text{Basis for } E_{2+8i} = \left\{ \begin{bmatrix} 1+8i \\ 5 \end{bmatrix} \right\}$$

c) Find a basis for the other eigenspace

$\lambda = 2-8i$:

$$\left[A - (2-8i)I \,|\, \vec{0} \right]$$

$$\left[\begin{array}{cc|c} 1+8i & -13 & 0 \\ 5 & -1+8i & 0 \end{array}\right]$$

$$\text{RREF} = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & \frac{-1+8i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$x_2 = t$$

$$x_1 + \left(\frac{-1+8i}{5}\right)x_2 = 0 \implies x_1 = -\left(\frac{-1+8i}{5}\right)x_2 \implies x_1 = \frac{1-8i}{5}t$$

$$\vec{x} = \begin{bmatrix} \frac{1-8i}{5} \\ 1 \end{bmatrix} t \qquad \text{Basis for } E_{2-8i} = \left\{ \begin{bmatrix} 1-8i \\ 5 \end{bmatrix} \right\}$$