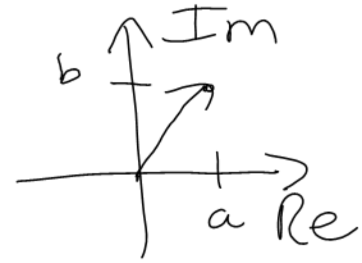
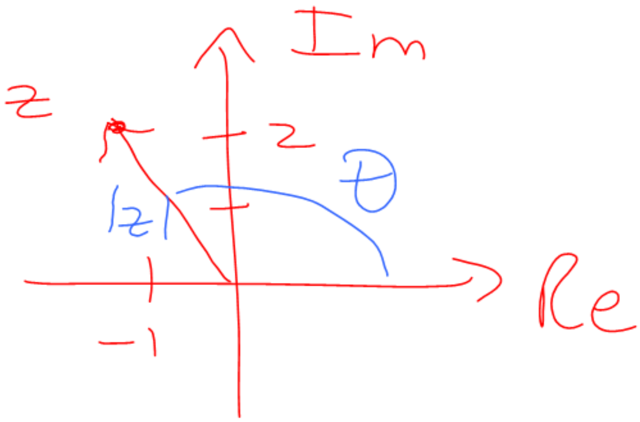
$z = a + bi$




**Definition:** The **length** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
The **principal argument** of $z = a + bi$ is the angle $\theta = \tan^{-1}(\frac{b}{a})$ $(+\pi?)$
We decide whether to add $\pi$ or not based on the graph of $z$.

**Example:** Let $z = -1 + 2i$. Graph $z$ then calculate $|z|$ and $\theta$.




$$|z| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

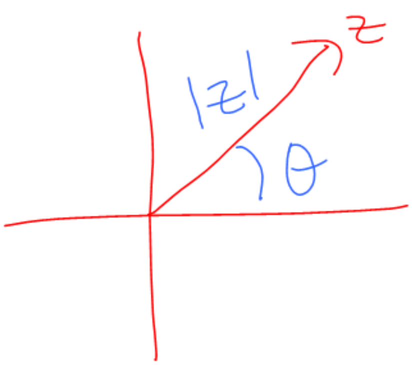
$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \ (+\pi?)$$

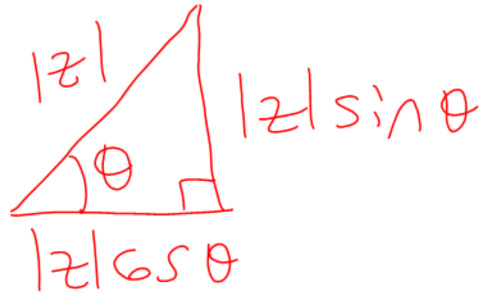
$$= \tan^{-1}\left(\frac{2}{-1}\right) + \pi$$

$$\approx 2.03 \text{ rads}$$

Add $\pi$ when real coefficient of $z$ is $< 0$

**Example:** Show that $z = |z|[\cos\theta + i\sin\theta]$.




$$z = |z|\cos\theta + i|z|\sin\theta$$
$$z = |z|[\cos\theta + i\sin\theta]$$

**Definition:** The **rectangular form** of a complex number is $z = a + bi$.
The **polar form** of a complex number is $z = |z|[\cos\theta + i\sin\theta]$.

$$\boxed{\begin{array}{c}\text{Phasor Notation}\\ |z|\angle\theta\end{array}}$$

**Example:** Express $z = -1 + 8i$ in polar form.

$$|z| = \sqrt{(-1)^2 + 8^2} = \sqrt{65}$$
$$\theta = \tan^{-1}\left(\frac{8}{-1}\right)\left(+\pi\,?\right)$$
$$= \tan^{-1}(-8) + \pi \checkmark$$
$$= -\tan^{-1}8 + \pi \checkmark$$
$$= \pi - \tan^{-1}8 \checkmark$$

$$z = |z|[\cos\theta + i\sin\theta]$$
$$z = \sqrt{65}\left[\cos(\pi - \tan^{-1}8) + i\sin(\pi - \tan^{-1}8)\right]$$

**Fact:** Multiplication and Division in Polar Form

Let $z_1 = |z_1|[\cos\theta_1 + i\sin\theta_1]$ and $z_2 = |z_2|[\cos\theta_2 + i\sin\theta_2]$.

Then $z_1 z_2 = |z_1|\,|z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ and

$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$.

**Comment:** When multiplying in polar form: multiply the lengths and add the angles. When dividing in polar form: divide the lengths and subtract the angles.

**Example:** Let $z_1 = 9 + 3\sqrt{3}i$ and $z_2 = 4\sqrt{3} - 12i$.
Find $\frac{z_1}{z_2}$ by converting to polar form.

$z_1$: $|z_1| = \sqrt{81+27} = \sqrt{108} = 6\sqrt{3}$

$\theta_1 = \tan^{-1}\left(\frac{3\sqrt{3}}{9}\right)$ ~~(+π?)~~

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$z_2$: $|z_2| = \sqrt{48+144} = \sqrt{192} = 8\sqrt{3}$

$\theta_2 = \tan^{-1}\left(\frac{-12}{4\sqrt{3}}\right)$ ~~(+π?)~~

$= \tan^{-1}(-\sqrt{3})$

$= -\frac{\pi}{3}$

Example Continued...

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

$$= \frac{6\sqrt{3}}{8\sqrt{3}}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{3}{4}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

polar form

$$= \frac{3}{4}i$$

rectangular form

**Fact:** De Moivre's Formula
Let $n$ be a positive integer.
If $z = |z|[\cos\theta + i\sin\theta]$ then $z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)]$.

When multiplying in polar form: lengths multiply, angles add

**Example:** Find $(1-i)^{21}$.

$$z = 1 - i$$

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) \quad \cancel{(+\pi?)}$$

$$= -\frac{\pi}{4}$$

$$z = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

$$z^{21} = \sqrt{2}^{21}\left[\cos\left(-\frac{21\pi}{4}\right) + i\sin\left(-\frac{21\pi}{4}\right)\right]$$

$$= \sqrt{2}^{20}\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$\boxed{-\frac{21\pi}{4} + \frac{24\pi}{4} = \frac{3\pi}{4}}$$

$$= 1024\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] \checkmark$$

$$= -1024 + 1024i \checkmark$$

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A complex number $z$ has 2 square roots
3 cube roots
⋮

FACT

$z = |z|[\cos\theta + i\sin\theta]$ has $n$ different $n^{th}$ roots $(n = 2, 3, \dots)$:

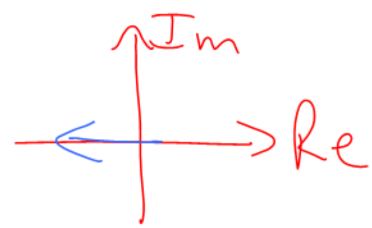
$$z^{1/n} = |z|^{1/n}\left[\cos\frac{\theta + 2\pi\alpha}{n} + i\sin\frac{\theta+2\pi\alpha}{n}\right] \quad (*)$$

where $\alpha = 0, 1, 2, \dots, n-1$

Ex: Find all the cube roots of $-1$.

$z = -1$

$|z| = 1 \qquad \theta = \pi$




From $(*)$ with $n = 3$

$$z^{1/3} = \underbrace{\cancel{|z|^{1/3}}}_{1}\left[\cos\frac{\theta+2\pi\alpha}{3} + i\sin\frac{\theta+2\pi\alpha}{3}\right]$$

$\alpha = 0, 1, 2$

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| $\alpha$ | $\frac{\theta + 2\pi\alpha}{3} = \frac{\pi + 2\pi\alpha}{3}$ |
|---|---|
| 0 | $\pi/3$ |
| 1 | $\pi$ |
| 2 | $5\pi/3$ |

$$z_1^{1/3} = 1\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_2^{1/3} = 1\left[\cos\pi + i\sin\pi\right] = -1$$

$$z_3^{1/3} = 1\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

To check:

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 = -1 \checkmark$$

$$(-1)^3 = -1 \checkmark$$

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = -1 \checkmark$$