Assignment due Mon Apr 8

Final Exam
Mon Apr 15
1:30 pm
TEC 174/175

Office Hours
Mon Apr 15 12-1pm

Exam Breakdown

15 Questions
Three Hours

Ch	% of Marks on Exam
1	18
2	21
3	23
4	13
5	11
7.3 and Complex	14

Example: Find the best-fit parabola through:

x	y
1	1
2	-2
3	3
4	4

$y = a_0 + a_1 x + a_2 x^2$

a_0, a_1, a_2 are the unknowns

$1(a_0) + x a_1 + x^2 a_2 = y$

$$\begin{array}{ccc} 1 & x & x^2 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{array}{c} y \\ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \end{array}$$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{20} \begin{bmatrix} 155 & -135 & 25 \\ -135 & 129 & -25 \\ 25 & -25 & 5 \end{bmatrix}$$

$[A|I] \sim [I|A^{-1}]$

or adjoint method

or MATLAB

Octave Online

$M = [4\ 10\ 30;$

$\ldots\ 30\ 100\ 354]$

format rat

inv(M)

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$= \frac{1}{20} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}}$$

$$= \frac{1}{20} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 6 \\ 22 \\ 84 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 60 \\ -72 \\ 20 \end{bmatrix} \begin{array}{l} \Leftarrow a_0 = 3 \\ \Leftarrow a_1 = -18/5 \\ \Leftarrow a_2 = 1 \end{array}$$

$$y = 3 - \frac{18}{5} x + x^2$$

Example: Find the best-fit curve $P = Ce^{kt}$ through:

t	P
0	5
1	8
3	12

$$\ln P = \ln\left(Ce^{kt}\right)$$

$$\ln P = \ln C + \ln e^{kt}$$

$$\ln P = \ln C + kt$$

k and $\ln C$ are the unknowns

$$1(\ln C) + tk = \ln P$$

$$\begin{array}{cc} 1 & t \\ \end{array} \qquad\qquad\qquad \ln P$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \ln C \\ k \end{bmatrix} = \begin{bmatrix} \ln 5 \\ \ln 8 \\ \ln 12 \end{bmatrix}$$

$$\vec{x}^{*} = (A^TA)^{-1}A^T\vec{b}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{14}\begin{bmatrix} 10 & -4 \\ -4 & 3 \end{bmatrix}$$

$$\vec{x}^{*} = \frac{1}{14}\underbrace{\begin{bmatrix} 10 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}}\begin{bmatrix} \ln 5 \\ \ln 8 \\ \ln 12 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 10 & 6 & -2 \\ -4 & -1 & 5 \end{bmatrix}\begin{bmatrix} \ln 5 \\ \ln 8 \\ \ln 12 \end{bmatrix}$$

Example Continued...

$$= \frac{1}{14}\begin{bmatrix} 10\ln 5 + 6\ln 8 - 2\ln 12 \\ -4\ln 5 - \ln 8 + 5\ln 12 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.69 \\ 0.279 \end{bmatrix} \begin{array}{l} \leftarrow \ln C \\ \leftarrow k \end{array}$$

$$k \approx 0.3$$

$$C = e^{\ln C} \approx e^{1.69} \approx 5$$

$$P = Ce^{kt}$$

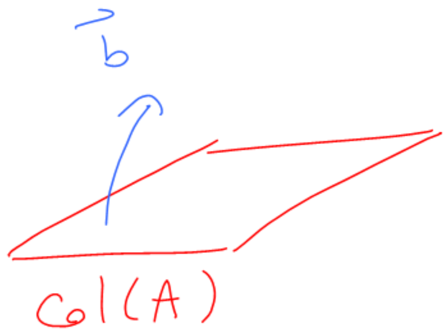
$$P = 5e^{0.3t}$$

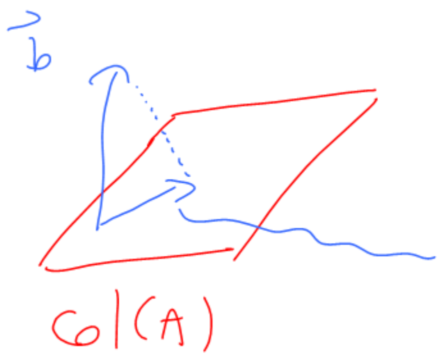
solvable

Recall that $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in col(A).
This follows from Sections 2.3 and 3.5.

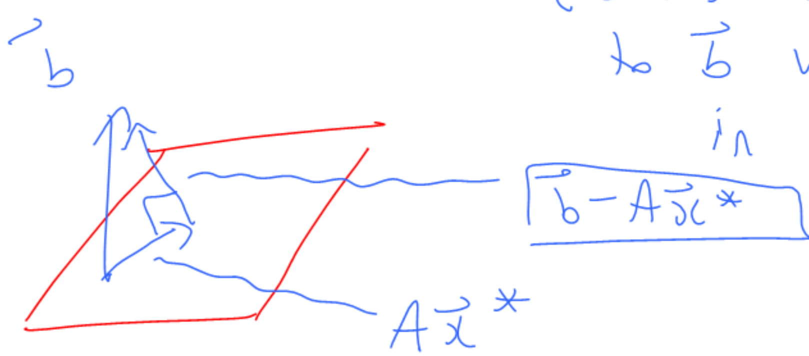
Recall that null(A^T)=[col(A)]$^{\perp}$. We saw this in Section 5.2.

Example: Derive the formula for $\vec{x}^*$ by considering an inconsistent system $A\vec{x} = \vec{b}$.





$A\vec{x}^*$ or $\text{Proj}_{\text{Col}(A)}\vec{b}$
(closest we can get to $\vec{b}$ while staying in Col(A))



$\vec{b} - A\vec{x}^*$ is in $[\text{Col}(A)]^{\perp}$

$\vec{b} - A\vec{x}^*$ is in null(A^T)

Example Continued...

$$A^T(\vec{b} - A\vec{x}^*) = \vec{0}$$

$$A^T\vec{b} - A^TA\vec{x}^* = \vec{0}$$

$$A^T\vec{b} = A^TA\vec{x}^*$$

$$A^TA\vec{x}^* = A^T\vec{b}$$

$$\cancel{(A^TA)^{-1}}A^TA\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

Complex Numbers

Definition: Let i be the **imaginary number** such that $i^2 = -1$.
If a and b are real numbers then $z = a + bi$ is a **complex number**.

Comment: The symbol i is sometimes written j. You may feel free to use either notation.

Example: Let $z_1 = -2 + 6i$ and $z_2 = 4 + 5i$. Calculate:

a) $-7z_1$

$= -7(-2+6i)$

$= 14 - 42i$

b) $z_1 + z_2$

$= 2 + 11i$

c) $z_1 - z_2$

$= -6 + i$

d) $z_1 z_2$

$= (-2+6i)(4+5i)$

$= -8 - 10i + 24i + \cancel{30i^2}$

-30

$= -38 + 14i$

$\boxed{i^2 = -1}$

Definition: The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$.

Example: Let $z = a + bi$. Show that $z\overline{z} = a^2 + b^2$.

$$z\overline{z} = (a+bi)(a-bi)$$
$$= a^2 - abi + abi \cancel{-b^2 i^2} + b^2$$
$$= a^2 + b^2 \leftarrow \text{real \#}$$

Example: Let $z_1 = 4 + 9i$ and $z_2 = -3 + 5i$. Calculate:

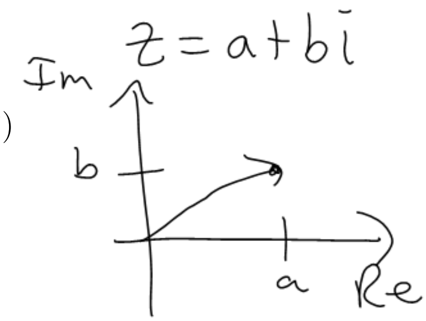
a) $\frac{1}{z_1}$

Multiply top and bottom by conjugate of denominator

$$= \frac{1}{(4+9i)} \cdot \frac{4-9i}{(4-9i)}$$
$$= \frac{4-9i}{97} \quad \leftarrow a^2+b^2$$
$$= \frac{4}{97} - \frac{9}{97}i$$

b) $\frac{z_1}{z_2}$

$$= \frac{(4+9i)}{(-3+5i)} \cdot \frac{(-3-5i)}{(-3-5i)}$$
$$= \frac{-12 - 20i - 27i \overset{+45}{\cancel{-45 i^2}}}{34} \quad \leftarrow a^2+b^2$$
$$= \frac{33 - 47i}{34} \quad \text{or} \quad \frac{33}{34} - \frac{47}{34}i$$



Definition: The **length** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
The **principal argument** of $z = a + bi$ is the angle $\theta = \tan^{-1}(\frac{b}{a})$ $(+\pi?)$
We decide whether to add π or not based on the graph of z.

Example: Let $z = -1 + 2i$. Graph z then calculate $|z|$ and θ.