We're going to recap the outer product expansion of AB from Section 3.1.

Example: Find $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ using the outer product expansion.

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 15 & 18 \end{bmatrix} + \begin{bmatrix} 14 & 16 \\ 28 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \checkmark$$

Example: Find $\begin{bmatrix} -1 & 9 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ using the outer product expansion.

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -3 \\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 18 & 9 \\ 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 6 \\ 14 & 9 \end{bmatrix} \checkmark$$

Definition: Let A be a symmetric $n \times n$ matrix.
Let $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$ be orthonormal eigenvectors written as columns.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
The **spectral decomposition of** A is:
$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \ldots + \lambda_n \vec{q}_n \vec{q}_n^{\,T}$

Example: Find a 3×3 matrix A with eigenvalues $\lambda = 2$ and $\lambda = 3$ so that:
$E_2 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix})$ and $E_3 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix})$.

Eigenvectors are orthogonal! ✓

$\vec{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{q}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

are orthonormal eigenvectors ✓

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \ldots$$

$$= \underbrace{2 \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}}}_{2/3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \overbrace{2 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}}^{1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

$$+ \underbrace{3 \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}}}_{1/2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$$

$$= \frac{4}{6}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{6}{6}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 13 & 1 & -2 \\ 1 & 13 & -2 \\ -2 & -2 & 16 \end{bmatrix} \leftarrow \text{symmetric}$$

Check: $A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ✓ $\quad A\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ✓ $\quad A\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ ✓

Example Continued...

Example: Suppose $Q^T AQ = D$. Solve for A then use the outer product expansion to derive the spectral decomposition.

$$Q^T A Q = D$$

Left-multiply by Q:

$$\underbrace{\cancel{QQ^T}}_{I} A Q = QD$$

$$AQ = QD$$

Right-multiply by Q^T:

$$A \underbrace{\cancel{QQ^T}}_{I} = QDQ^T$$

$$A = QDQ^T$$

$$= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{q}_1 & \lambda_2 \vec{q}_2 & \cdots & \lambda_n \vec{q}_n \end{bmatrix} \begin{bmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix}$$

$$= \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \ldots + \lambda_n \vec{q}_n \vec{q}_n^{\,T}$$

Appendix: Other Topics

7.3 Least Squares Approximation

(Curve Fitting)

Recall that a system $A\vec{x} = \vec{b}$ may be inconsistent.

unsolvable

$$\boxed{\begin{array}{l} A\vec{s} \approx \vec{b} \\ \vec{b} - A\vec{s} \approx \vec{0} \end{array}}$$

Definition: Given an approximate solution $\vec{s}$, the **error vector** is $\vec{b} - A\vec{s}$ and the **error** is $||\vec{b} - A\vec{s}||$.

Definition: The **least squares solution** $\vec{x}^*$ is the approximate solution with the minimum error.

Comment: Recall that $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$. The terminology **least squares solution** emphasizes that we're making the length of the error vector as small as possible.

Fact: The **least squares solution** to a system $A\vec{x} = \vec{b}$ is $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$.

Comment: We'll assume that the columns of A are linearly independent so that $(A^T A)^{-1}$ exists.

Example: The system $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ is inconsistent.

Find the least squares solution $\vec{x}^*$.

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 6 \\ 9 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Example: Calculate the error for $\vec{x}^*$ above. What can you say about the error for any other vector $\vec{x}$?

error vector $\vec{b} - A\vec{x}^* = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

error $= \left\| \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{3}$

For any other $\vec{x}$, error $> \sqrt{3}$.

Example: Find the best-fit line $y = a_0 + a_1 x$.
The best-fit line is also called the **least squares regression line**.

x	y
0	4
1	1
2	0

a_0 and a_1 are the unknowns.

$$a_0 + a_1 x = y$$

$$1(a_0) + x(a_1) = y$$

System
$$\begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{array}{c} y \\ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

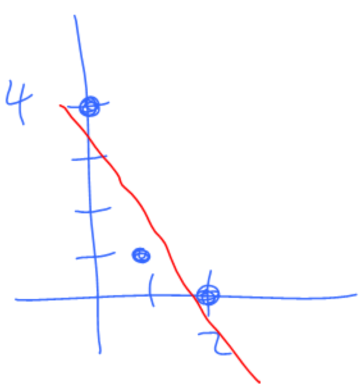
...

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

...

$$= \frac{1}{6} \begin{bmatrix} 22 \\ -12 \end{bmatrix} \text{ or } \begin{bmatrix} 11/3 \\ -2 \end{bmatrix} \begin{array}{l} \Leftarrow a_0 \\ \Leftarrow a_1 \end{array}$$



$$y = a_0 + a_1 x$$

$$y = \frac{11}{3} - 2x$$