Definition: Let A be a matrix with linearly independent columns.
The **QR Factorization of** A is:
$A = QR$ where Q is an orthogonal matrix and R is upper triangular.

orthonormal columns

Example: Let $A = QR$ for an orthogonal matrix Q. Show that $R = Q^T A$.

$$A = QR$$

Left-multiply by Q^T: $\quad Q^T A = \underbrace{Q^T Q}_{I} R$

$$Q^T A = R$$

$$R = Q^T A$$

Fact: Let $A = QR$ for an orthogonal matrix Q.
To find Q: Apply Gram-Schmidt to the columns of A, and normalize.
Then $R = Q^T A$.

Example: Find Q and R for $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

Q: Gram-Schmidt on columns of A, and normalize.

Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \mathrm{proj}_X \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \mathrm{proj}_X \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \mathrm{proj}_{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \mathrm{proj}_{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{0}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$5\vec{v}_3 = 5 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ -1 \\ 15 \end{bmatrix}$$

Orthogonal Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 15 \end{bmatrix} \right\}$

Example Continued...

Orthonormal Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{230}}\begin{bmatrix} 0 \\ 2 \\ -1 \\ 15 \end{bmatrix} \right\}$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{230} \\ 0 & 2/\sqrt{5} & -1/\sqrt{230} \\ 0 & 0 & 15/\sqrt{230} \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 2/\sqrt{230} & -1/\sqrt{230} & 15/\sqrt{230} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{5}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ 0 & 0 & \frac{46}{\sqrt{230}} \end{bmatrix}$$

Example: Approximating the eigenvalues of A. **This example will not be tested**.

Consider the following procedure:
Find $A = Q_0 R_0$.
Let $A_1 = R_0 Q_0$ then find $A_1 = Q_1 R_1$.
Let $A_2 = R_1 Q_1$ then find $A_2 = Q_2 R_2$ etc.
Each matrix A_k has the same eigenvalues as A.
As $k \to \infty$, A_k becomes upper triangular.

Suppose we start with matrix A and produce $A_4 = \begin{bmatrix} 1.98 & 2.52 \\ 0.03 & 7.01 \end{bmatrix}$.

a) Does A_4 have the same eigenvalues as A?

Yes

b) Is A_4 approximately upper triangular?

$0.03 \approx 0$ Yes

c) Estimate the eigenvalues of A.

$\lambda \approx 1.98, 7.01$

5.4 Orthogonal Diagonalization

Recall that if Q is orthogonal then $Q^{-1} = Q^T$.

Definition: An $n \times n$ matrix A is **orthogonally diagonalizable** if there exist an orthogonal matrix Q and a diagonal matrix D so that $Q^T AQ = D$.

(Compare with Section 4.4 $P^{-1}AP = D$)

Fact: Let A be an $n \times n$ matrix. The matrix A is orthogonally diagonalizable if and only if A is symmetric.

$A^T = A$

Example: Is A orthogonally diagonalizable?

a) $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

$A^T = A$ ✓

Yes

b) $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

$A^T \neq A$

No

Example: The matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ has eigenvalues $\lambda = 4$ and $\lambda = 7$.
Find Q that orthogonally diagonalizes A.

Q: Find an $\underline{\text{orthonormal}}$ basis for each eigenspace.

$\lambda = 7$: $[A - 7I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \rightsquigarrow \\ \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{ RREF}$$

$x_3 = t$

$x_1 - x_3 = 0 \implies x_1 = t$

$x_2 - x_3 = 0 \implies x_2 = t$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$ Basis for $E_7 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Orthonormal Basis for $E_7 = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\lambda = 4$: $[A - 4I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Example Continued...

$$\rightsquigarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$x_2 = s \quad x_3 = t$

$x_1 + x_2 + x_3 = 0 \implies x_1 = -s - t$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t \qquad \text{Basis for } E_4 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Gram-Schmidt

Partial Basis $X = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \text{proj}_X \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$2\vec{v}_2 = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Orthogonal Basis for E_4 $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

Orthonormal Basis for $E_4 = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

To check:

$$Q^T A Q = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$