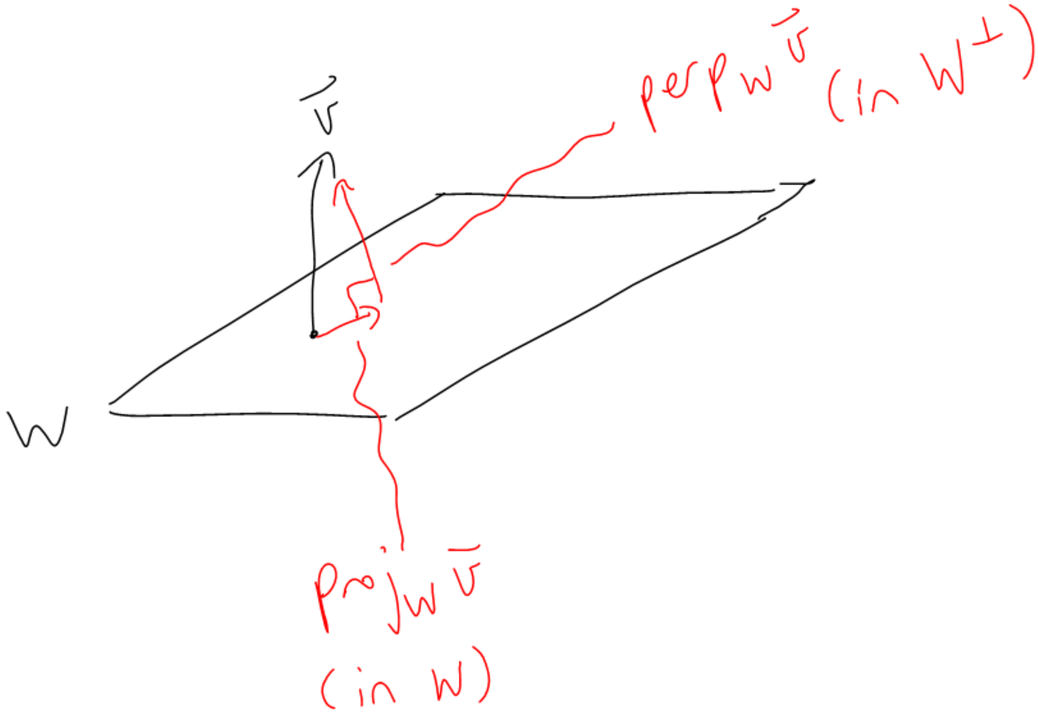
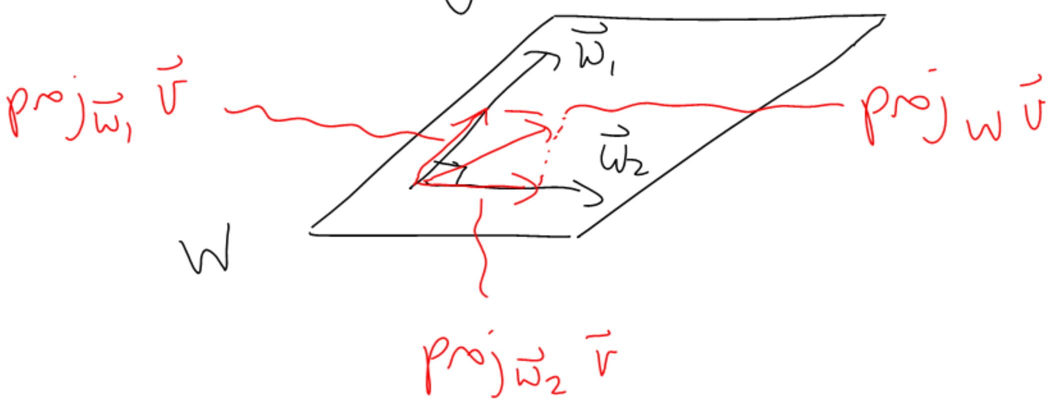


Given an orthogonal basis for W: $\{\vec{w}_1, \vec{w}_2\}$



$$\text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v}$$

5.3 The Gram-Schmidt Procedure

Example: Let $W = \text{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix}\right)$. Find an orthogonal basis for W.

$\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ Partial Basis $X = \left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$

$\vec{v}_2 = \begin{bmatrix}1\\2\\4\\5\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\2\\4\\5\end{bmatrix}$

$\boxed{\text{Subtracts off the part of } \begin{bmatrix}1\\2\\4\\5\end{bmatrix} \text{ that is parallel to } \begin{bmatrix}1\\1\\1\\1\end{bmatrix}.}$

$= \begin{bmatrix}1\\2\\4\\5\end{bmatrix} - \cancelto{3}{\frac{12}{4}} \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$= \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$ Partial Basis $X = \left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}\right\}$

$\vec{v}_3 = \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix}$

$= \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \text{proj}_{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \text{proj}_{\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}} \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix}$

$= \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\|^2} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \cdots$

$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a}$

Example Continued...

$$= \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} \overset{+2}{\cancel{-\frac{(-8)}{4}}} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{(-7)}{10}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$$

Can scale these vectors

$$10\vec{v}_3 = 10\begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} + 20\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 7\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}16\\-17\\-13\\14\end{bmatrix}$$

Orthogonal basis for $W = \left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}16\\-17\\-13\\14\end{bmatrix} \right\}$

Comment: This procedure is called **Gram-Schmidt Orthogonalization**.

Example: Modify the basis above to create an orthonormal basis for W.

$$\left\{ \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}, \frac{1}{\sqrt{910}}\begin{bmatrix}16\\-17\\-13\\14\end{bmatrix} \right\}$$

Example: Find an orthogonal basis for $\mathbb{R}^3$ containing $\begin{bmatrix}1\\1\\5\end{bmatrix}$.

Start with any basis for $\mathbb{R}^3$ containing $\begin{bmatrix}1\\1\\5\end{bmatrix}$.

Say $\left\{ \begin{bmatrix}1\\1\\5\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$

It's a basis for $\mathbb{R}^3$ because

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 0 & 0 \end{vmatrix} \neq 0$$

(Fundamental Theorem of Invertible Matrices Section 3.5)

Gram-Schmidt:

Partial Basis $X = \left\{ \begin{bmatrix}1\\1\\5\end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{27}\begin{bmatrix}1\\1\\5\end{bmatrix}$$

$$27\vec{v}_2 = 27\begin{bmatrix}1\\0\\0\end{bmatrix} - 1\begin{bmatrix}1\\1\\5\end{bmatrix}$$

$$= \begin{bmatrix}26\\-1\\-5\end{bmatrix}$$

Partial Basis $X = \left\{ \begin{bmatrix}1\\1\\5\end{bmatrix}, \begin{bmatrix}26\\-1\\-5\end{bmatrix} \right\}$

Example Continued...

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{27} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - \frac{(-1)}{702} \begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}$$

$$702\,\vec{v}_3 = 702 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 26 \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 675 \\ -135 \end{bmatrix}$$

Orthogonal basis for $\mathbb{R}^3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 675 \\ -135 \end{bmatrix} \right\}$