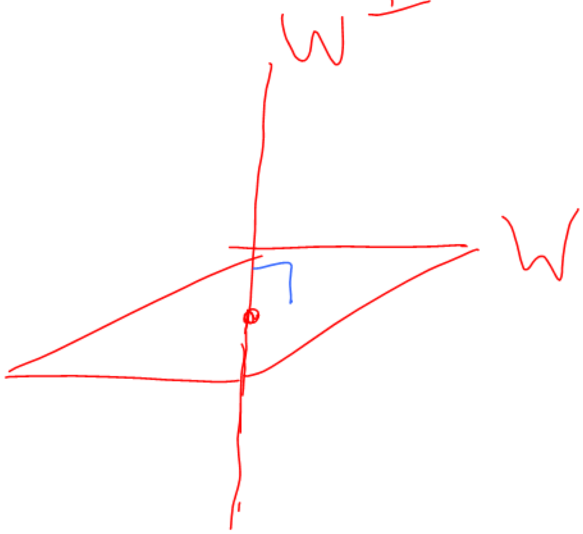
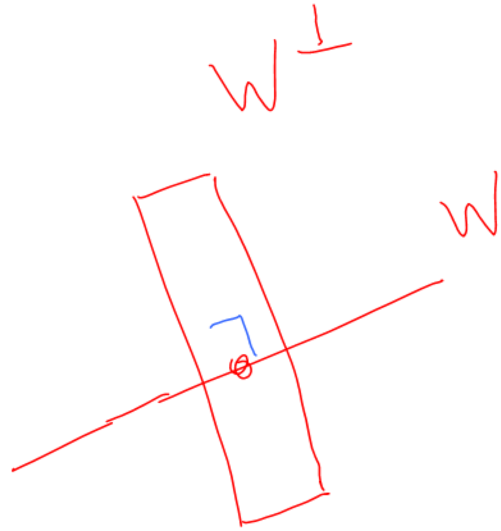
In $\mathbb{R}^3$





Example: Let $W = \{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $3x + y = 0\}$. Find a basis for W and for $W^{\perp}$.

W: line in $\mathbb{R}^2$

Two points $P = (0,0)$ $Q = (1,-3)$

$$\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Basis for $W = \{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\}$

$W^{\perp}$:

$$\begin{array}{cc} x & y \\ \multicolumn{2}{c}{\left[\begin{array}{cc|c} 1 & -3 & 0 \end{array}\right]} \end{array}$$

RREF

$y = t$

$x - 3y = 0 \Rightarrow x = 3t$

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} t$$

Basis for $W^{\perp} = \{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 6 \end{bmatrix}$ and let $W = \text{row}(A)$. Find a basis for $W^\perp$.

$W = \text{Span}([1\ 0\ 0\ 4], [0\ 1\ 1\ 6])$

$W^\perp$:

$$\begin{array}{cccc} w & x & y & z \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & 6 & 0 \end{array}\right] \quad \text{RREF}$$

$y = s$

$z = t$

$w + 4z = 0 \Rightarrow w = -4t$

$x + y + 6z = 0 \Rightarrow x = -s - 6t$

$$\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ -6 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $W^\perp = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right\}$

More Formally: $W^\perp = \{\vec{x} \mid \vec{x}$ is orthogonal to each row of $A\}$

Solve $A\vec{x} = \vec{0}$

$W^\perp = \text{null}(A)$

Fact: For any matrix A, $[\text{row}(A)]^\perp = \text{null}(A)$.

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$ and let $W = \text{col}(A)$. Find a basis for $W^\perp$.

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right)$$

$W^\perp$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$z = t$

$x = 0$

$y = 0$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $W^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

More formally:

$W = \text{Col}(A)$

$= \text{row}(A^T)$

$W^\perp = \text{null}(A^T)$

Fact: For any matrix A, $[\text{col}(A)]^\perp = \text{null}(A^T)$.

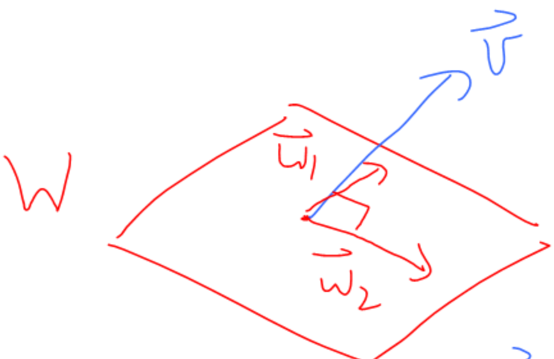
We'll revisit this in Section 7.3

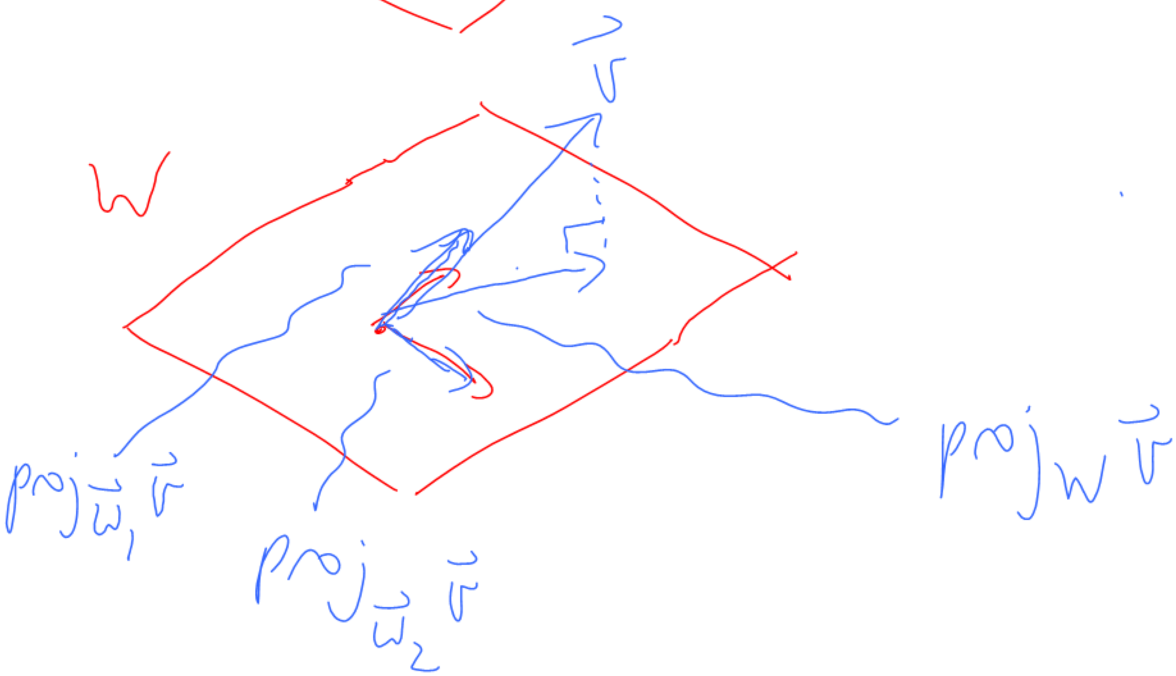
Definition: Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$. The **orthogonal projection of** $\vec{v}$ onto W is:

$\text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v} + \ldots + \text{proj}_{\vec{w}_k} \vec{v}$.

Comment: This formula only applies when the basis for W is **orthogonal**.

Example: Let W be a plane through the origin in $\mathbb{R}^3$. Let $\vec{v}$ be a vector in $\mathbb{R}^3$ that does not lie in W. Sketch W, $\vec{v}$ and $\text{proj}_W \vec{v}$.





$\text{proj}_W \vec{v}$ is as close as we can get to $\vec{v}$ while staying on W.

Example: W has orthogonal basis $\mathcal{B} = \{ \overset{\vec{w}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \overset{\vec{w}_2}{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}} \}$.

Find the orthogonal projection of $\vec{v} = \begin{bmatrix} 1 \\ 5 \\ -3 \\ 7 \end{bmatrix}$ onto W.

Orthogonal basis

$$\Rightarrow \operatorname{proj}_W \vec{v} = \operatorname{proj}_{\vec{w}_1} \vec{v} + \operatorname{proj}_{\vec{w}_2} \vec{v}$$

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\|\vec{w}_1\|^2} \vec{w}_1 + \ldots$$

$$= \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{8}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ -4 \\ 0 \end{bmatrix}$$

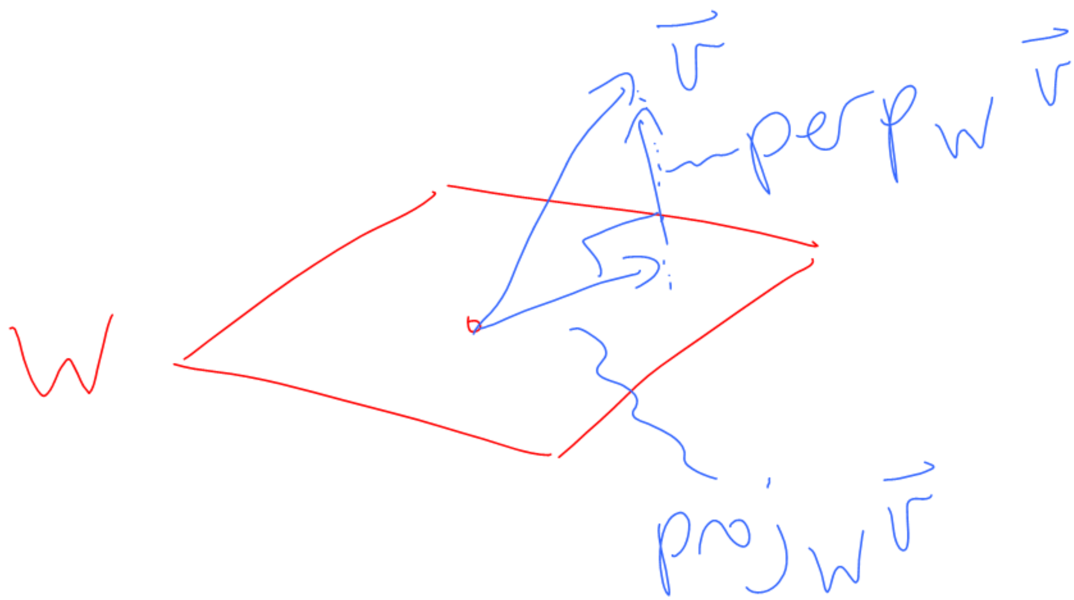
Definition: The **orthogonal decomposition** of $\vec{v}$ with respect to W is:

$\vec{v} = \text{proj}_W\vec{v} + \text{perp}_W\vec{v}$

where $\text{proj}_W\vec{v}$ is in W and $\text{perp}_W\vec{v}$ is in $W^\perp$.

"perp with respect to W of v"

Example: Let W be a plane through the origin in $\mathbb{R}^3$. Let $\vec{v}$ be a vector in $\mathbb{R}^3$ that does not lie in W. Sketch W, $\vec{v}$, $\text{proj}_W\vec{v}$ and $\text{perp}_W\vec{v}$.



Example: W has orthogonal basis $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \}$.

$\vec{w}_1$

$\vec{w}_2$

Find the orthogonal decomposition of $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ with respect to W.

Find $\text{proj}_W \vec{v}$ and $\text{perp}_W \vec{v}$.

Orthogonal basis

$$\Rightarrow \text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v}$$

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\|\vec{w}_1\|^2} \vec{w}_1 + \ldots$$

$$= \frac{-4}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{11}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{-18}{9} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{11}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 4 \\ -11 \\ 40 \end{bmatrix}$$

$$\text{proj}_W \vec{v} + \text{perp}_W \vec{v} = \vec{v}$$

$$\text{perp}_W \vec{v} = \vec{v} - \text{proj}_W \vec{v}$$

$$= \frac{9}{9} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 4 \\ -11 \\ 40 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 5 \\ 20 \\ 5 \end{bmatrix}$$