TEST REVIEW

(1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ has RREF $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ has RREF $= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Find a basis for:

a) Col(A)

b) row(A), consisting of rows of A

c) null(A)

a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Use columns 1 and 2 of A

b) $\text{row}(A) = \text{Col}(A^T)$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Use columns 1 and 3 of A^T

or $\{ [1\ 1\ 1], [1\ 0\ 0] \}$

c) Solve $A\vec{x} = \vec{0}$

$[A \mid \vec{0}]$

$[\text{RREF of } A \mid \vec{0}]$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$z = t$$
$$x = 0$$
$$y + z = 0 \Longrightarrow y = -t$$

$$\text{null}(A) \longrightarrow \vec{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t$$

$$\text{Basis for null}(A) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

② Compute $\det A$ for $A = \begin{bmatrix} 1 & 2 & -3 \\ 9 & 8 & 7 \\ 4 & -5 & 4 \end{bmatrix}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ & & \end{bmatrix}$$

$$|A| = 1 \begin{vmatrix} 8 & 7 \\ -5 & 4 \end{vmatrix} - 2 \begin{vmatrix} 9 & 7 \\ 4 & 4 \end{vmatrix} - 3 \begin{vmatrix} 9 & 8 \\ 4 & -5 \end{vmatrix}$$
$$= 1(67) - 2(8) - 3(-77)$$
$$= 282$$

OR

$$\begin{array}{ccccc} & & - & - & - \\ 1 & 2 & -3 & 1 & 2 \\ 9 & 8 & 7 & 9 & 8 \\ 4 & -5 & 4 & 4 & -5 \end{array}$$

96 35 −72 32 56 135

$$|A| = 96 + 35 - 72 + 32 + 56 + 135 = 282$$

(3) Solve using Cramer's Rule

$$\begin{cases} 4x - 3y = 12 \\ 8x + 6y = 16 \end{cases}$$

$$|A| = \begin{vmatrix} 4 & -3 \\ 8 & 6 \end{vmatrix} = 48$$

$$|A_1| = \begin{vmatrix} 12 & -3 \\ 16 & 6 \end{vmatrix} = 120$$

$$|A_2| = \begin{vmatrix} 4 & 12 \\ 8 & 16 \end{vmatrix} = -32$$

$$x = \frac{|A_1|}{|A|} = \frac{120}{48} = \frac{5}{2} \qquad y = \frac{|A_2|}{|A|} = \frac{-32}{48} = \frac{-2}{3}$$

(4) a) Find all the eigenvalues of $A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$-6-6\lambda+\lambda+\lambda^2+12=0$$

$$\lambda^2-5\lambda+6=0$$

$$(\lambda-2)(\lambda-3)=0$$

$$\lambda=2,3$$

b) Find all eigenvectors of $A=\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$ corresponding to $\lambda=3$.

Solve $[A-\lambda I \mid \vec{0}]$

$[A-3I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right]$$

$$\frac{R_1}{3} \quad \left[\begin{array}{cc|c} 1 & -\frac{4}{3} & 0 \\ 3 & -4 & 0 \end{array}\right]$$

$$R_2 - 3R_1 \quad \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$$x_2 = t$$

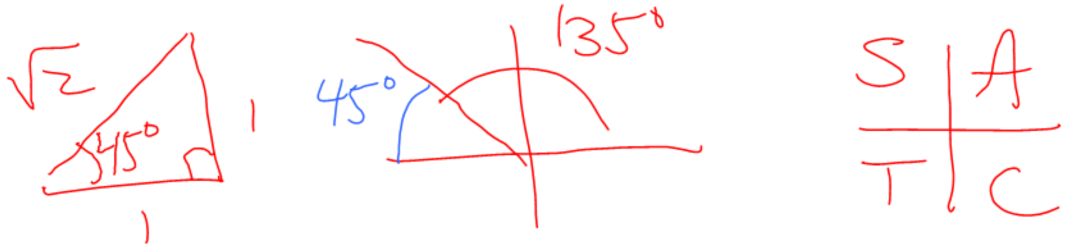
$$x_1 - \frac{4}{3}x_2 = 0 \Rightarrow x_1 = \frac{4}{3}t$$

$$\vec{x} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} t \qquad (t \neq 0)$$

OR $$\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} t \qquad (t \neq 0)$$

⑤ Find the standard matrix for $T: \mathbb{R}^2 \to \mathbb{R}^2$ where T first rotates a vector by $135°$ then projects it on the line through the origin with $\vec{d} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$[T_1] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 135°$$



$$\cos 135° = \frac{-1}{\sqrt{2}} \qquad \sin 135° = \frac{1}{\sqrt{2}}$$

$$[T_1] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[T_2] = \frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$[T] = [T_2][T_1]$$

$$= \frac{1}{5\sqrt{2}}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{5\sqrt{2}}\begin{bmatrix} -6 & -2 \\ 3 & 1 \end{bmatrix}$$

6 a) Find the LU Factorization of $A = \begin{bmatrix} 3 & 2 & -4 \\ -27 & -12 & 43 \\ 21 & 26 & -22 \end{bmatrix}$

$$\begin{matrix} R_2 + 9R_1 \\ R_3 - 7R_1 \end{matrix} \quad \begin{bmatrix} 3 & 2 & -4 \\ 0 & 6 & 7 \\ 0 & 12 & 6 \end{bmatrix} \begin{matrix} \\ k=-9 \\ k=7 \end{matrix}$$

$$R_3 - 2R_2 \qquad \underbrace{\begin{bmatrix} 3 & 2 & -4 \\ 0 & 6 & 7 \\ 0 & 0 & -8 \end{bmatrix}}_{U} \quad k=2$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -9 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$$

b) Solve $\left[\begin{array}{ccc|c} 3 & 2 & -4 & -5 \\ -27 & -12 & 43 & 78 \\ 21 & 26 & -22 & 7 \end{array}\right]$

using the LU factorization above.

$$A\vec{x} = \vec{b}$$

$$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$

1) $L\vec{y} = \vec{b}$

$$\begin{array}{ccc} y_1 & y_2 & y_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ -9 & 1 & 0 & 78 \\ 7 & 2 & 1 & 7 \end{array}\right] \qquad \begin{aligned} y_1 &= -5 \\ y_2 &= 33 \\ y_3 &= -24 \end{aligned}$$

2) $U\vec{x} = \vec{y}$

$$\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\
\left[\begin{array}{ccc|c} 3 & 2 & -4 & -5 \\ 0 & 6 & 7 & 33 \\ 0 & 0 & -8 & -24 \end{array}\right]
\end{array}
\quad
\begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array}$$