Definition: An **orthogonal matrix** Q is an $n \times n$ matrix whose columns form an orthonormal set. For example, the following matrix is orthogonal:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

!

$$\left\| \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

Fact: A square matrix Q is orthogonal if and only if $Q^T Q = I$.

Example: Verify that $Q^T Q = I$ for $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$Q^T Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

ones indicate that columns of Q have length 1

zeros indicate that columns of Q are orthogonal

Fact: If Q is orthogonal then $Q^{-1} = Q^T$.

Example: Prove the fact above.

Q orthogonal

$\Rightarrow \quad Q^T Q = I$

Right-multiply by Q^{-1}:

$$Q^T \underbrace{\cancel{Q Q^{-1}}}_{I} = I Q^{-1}$$

$$Q^T = Q^{-1}$$

Example: Let Q be an orthogonal matrix. Show that Q^{-1} is orthogonal.

To show that M is orthogonal: $M^T M = I$

" Q^{-1} " : $(Q^{-1})^T Q^{-1} = I$

$$(Q^{-1})^T Q^{-1} = (Q^T)^T Q^{-1} \quad (Q \text{ is orthogonal})$$

$$= Q Q^{-1}$$

$$= I \checkmark$$

Example: Determine all values of x, y and z so that $\begin{bmatrix} \frac{1}{2} & y \\ x & z \end{bmatrix}$ is an orthogonal matrix.

$$\left\| \begin{bmatrix} \frac{1}{2} \\ x \end{bmatrix} \right\| = 1$$

$$\sqrt{\frac{1}{4} + x^2} = 1$$

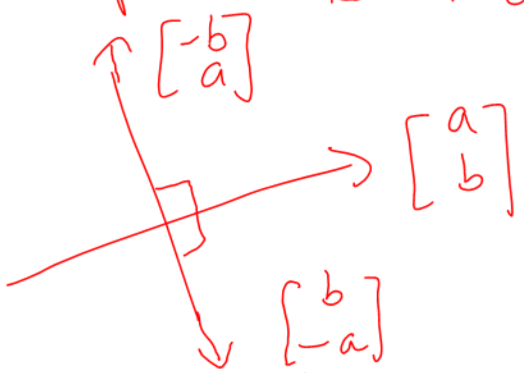
$$\frac{1}{4} + x^2 = 1$$

$$x^2 = \frac{3}{4}$$

$$x = \pm\sqrt{\frac{3}{4}}$$

$$x = \pm\frac{\sqrt{3}}{2}$$

If 1st column is $\begin{bmatrix} a \\ b \end{bmatrix}$ then there are 2 options for the 2nd column:



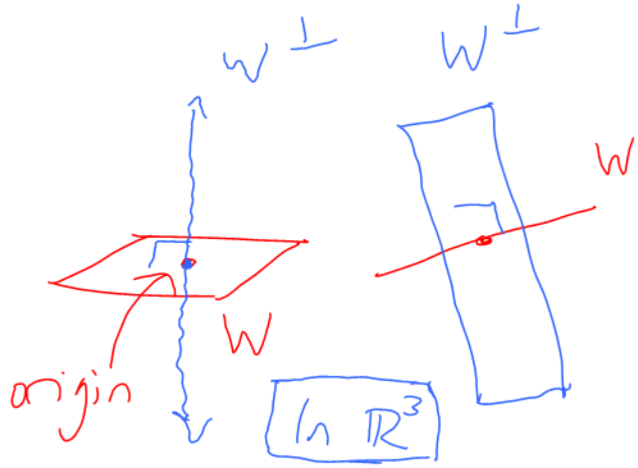
(all vectors have length 1)

$x = \frac{\sqrt{3}}{2}$: $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ $\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

$x = \frac{-\sqrt{3}}{2}$: $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$ $\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$

5.2 Orthogonal Complements and Projections

Throughout Chapter 5, W will represent a subspace of $\mathbb{R}^n$. Rephrased: W is the span of one or more vectors in $\mathbb{R}^n$.



Definition: The **orthogonal complement of** W is:
$W^\perp = \{\vec{v}$ in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w}$ in $W\}$.
$W^\perp$ is pronounced "W perp".

Example: Let $W = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}\right)$. Find $W^\perp$.

$$W^\perp = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ such that } \begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = 0 \text{ and } \begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}1\\1\\3\end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ such that } x = 0 \text{ and } x + y + 3z = 0 \right\}$$

Solve
$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right]$$

shortcut

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \text{ RREF}$$

$z = t$

$x = 0$

$y + 3z = 0 \Rightarrow y = -3t$

$\vec{x} = \begin{bmatrix}0\\-3\\1\end{bmatrix} t$ $\quad W^\perp = \text{span}\left(\begin{bmatrix}0\\-3\\1\end{bmatrix}\right)$ $\quad$ A basis for $W^\perp = \left\{\begin{bmatrix}0\\-3\\1\end{bmatrix}\right\}$

Recall that the **dimension** of a subspace W is the number of vectors in a basis for W.

(Section 3.5)

Three Facts about $W^\perp$

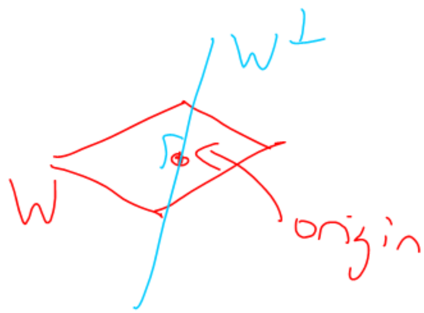
For any subspace W of $\mathbb{R}^n$:

1) $\dim W + \dim W^\perp = n$

2) $W \cap W^\perp = \{\vec{0}\}$

3) $(W^\perp)^\perp = W$

In $\mathbb{R}^3$



Example: Let $W = \text{span}\left(\underbrace{\begin{bmatrix}1\\4\\3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}}_{\text{not multiples}}\right)$. Find the dimension of W and $W^\perp$.

$\dim W = 2$

$\dim W + \dim W^\perp = 5$

$\dim W^\perp = 3$