4.4 Diagonalization

Definition: An $n \times n$ matrix A is **diagonalizable** if there exist an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

Fact: To find P we find a basis for each eigenspace of A. The basis vectors go into the columns of P. The matrix D has the eigenvalues on the diagonal, in the same order as P.

Example: Let $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find P and D that diagonalize A. 1=2,3,3 (A is upper triangular) [A-2I10] $\chi_{2}=0, \chi_{3}=0$

Example Continued...

$$\lambda = 3 : \left[A - 3I \right] 0$$

$$\left[-10 - 2 \right] 0$$

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Fact: A is diagonalizable if and only if: geometric multiplicity=algebraic multiplicity for all eigenvalues of A.

Example: Diagonalize $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$ (if possible). A = 4, A = 4 (A is lower triangular) A = 4: A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | O | A = 4 [O | O | A = 4

Example: Let $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$. Find the characteristic equation, the algebraic multiplicity of $\lambda = 4$ and the geometric multiplicity of $\lambda = 4$. Explain, in terms of algebraic and geometric multiplicity, why A can't be diagonalized.

characteristic equation $|A-\lambda I| = 0$ $|4-\lambda| = 0$ $|4-\lambda|^2 = 0$ algebraic multiplicity of $\lambda = 4$ is 2 geometric $|A-\lambda| = 0$ $|A-\lambda| = 0$ |A- **Fact:** Let n be a positive integer. If D is diagonal then D^n is diagonal, with n-th powers on the diagonal.

Example: Calculate
$$\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}^2$$
.

$$= \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (-4) & 0 \\ 0 & 3^2 \end{bmatrix}$$

Fact: Let n be a positive integer. If $P^{-1}AP = D$ then $A^n = PD^nP^{-1}$.

Example: Prove the fact above.

$$P^{-1}AP = D$$
 $PP^{-1}AP = PD$
 $APP^{-1} = PDP^{-1}PDP^{-1}$
 $A^{n} = PDP^{-1}PDP^{-1}PDP^{-1}$
 $= PD^{n}P^{-1}$

Example:
$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
 diagonalizes A to produce $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Find A^k , where k is a positive integer.

$$P^{-1}AP = D$$

$$AP = PD$$

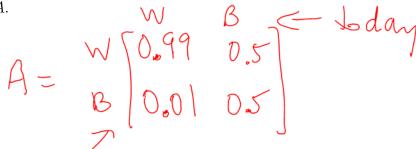
$$A = PDP^{-1}$$

$$A^{k} = PDP^{-1} ... PDP^{-1}$$

$$= PD^{k}P^{-1}$$

Example: Application of A^n and eigenvectors. This example will not be tested. Consider a company with 1000 machines.

a) Suppose a working machine has a 99% probability of working tomorrow. Suppose a broken machine has a 50% probability of being broken tomorrow. Write down the probability matrix, A.



b) Suppose all machines are working today. Write down the initial state vector, \vec{v} .

c) How many machines will be working or broken tomorrow?

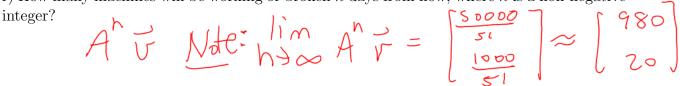
d) How many machines will be working or broken two days from now?

$$A^2 \vec{v} \simeq \begin{bmatrix} 985\\15 \end{bmatrix}$$

e) How many machines will be working or broken three days from now?

$$A^3 V \approx \begin{bmatrix} 9837 \\ 17 \end{bmatrix}$$

f) How many machines will be working or broken n days from now, where n is a non-negative



g) What initial state vector \vec{v} would have $A\vec{v} = \vec{v}$? This is called the **steady-state vector** because the state after one day is the same as the initial state.

