4.4 Diagonalization

Definition: An $n \times n$ matrix A is **diagonalizable** if there exist an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

Fact: To find P we find a basis for each eigenspace of A. The basis vectors go into the columns of P. The matrix D has the eigenvalues on the diagonal, in the same order as P.

Example: Let $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find P and D that diagonalize A.

$\lambda = 2, 3, 3$ (A is upper triangular)

$\lambda = 2$:

$$[A - \lambda I \mid \vec{0}]$$

$$[A - 2I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \rightsquigarrow \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{RREF}$$

$x_1 = t$

$x_2 = 0, \quad x_3 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$$

Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Example Continued...

$\lambda = 3$: $[A - 3I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{ RREF}$$

$x_2 = s$ $\quad x_3 = t$

$x_1 + 2x_3 = 0 \Rightarrow x_1 = -2t$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $E_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ basis vectors in columns, in any order

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ eigenvalues, in same order as P.

To Check: $P^{-1} A P = D$

(P^{-1} exists when P exists)

Fact: A is diagonalizable if and only if:
geometric multiplicity=algebraic multiplicity for all eigenvalues of A.

Example: Diagonalize $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$ (if possible).

$\lambda = 4, 4$ (A is lower triangular)

$\lambda = 4$: $[A - 4I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{cc} x_1 & x_2 \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_2 = t$

$x_1 = 0$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$ Basis for $E_4 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Not enough basis vectors $\Rightarrow$ can't diagonalize A.

Example: Let $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$. Find the characteristic equation, the algebraic multiplicity of $\lambda = 4$ and the geometric multiplicity of $\lambda = 4$. Explain, in terms of algebraic and geometric multiplicity, why A can't be diagonalized.

characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)^2 = 0$$

algebraic multiplicity of $\lambda = 4$ is 2
geometric " " is 1

geo. mult. < alg. mult. $\Rightarrow$ can't diagonalize A

Fact: Let n be a positive integer. If D is diagonal then D^n is diagonal, with n-th powers on the diagonal.

Example: Calculate $\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}^2$.

$$= \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (-4)^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

Fact: Let n be a positive integer. If $P^{-1}AP = D$ then $A^n = PD^nP^{-1}$.

Example: Prove the fact above.

$$P^{-1}AP = D$$

$$\cancel{PP^{-1}}AP = PD$$

$$A\cancel{PP^{-1}} = PDP^{-1}$$

$$A^n = \underbrace{PD\cancel{P^{-1}}}\underbrace{\cancel{P}D\cancel{P^{-1}}}\underbrace{\cancel{P}D\cancel{P^{-1}}} \cdots \underbrace{\cancel{P}DP^{-1}}$$

$$= PD^nP^{-1}$$

Example: $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizes A to produce $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Find A^k, where k is a positive integer.

$$P^{-1}AP = D$$

$$AP = PD$$

$$A = PDP^{-1}$$

$$A^k = PDP^{-1} \cdots PDP^{-1}$$

$$= PD^kP^{-1}$$

$$P^{-1}: \quad [P \mid I] \rightsquigarrow [I \mid P^{-1}]$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D^k = \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

$$A^k = PD^kP^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & 0 & 4^k \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & 0 & 4^k - 3^k \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

Example: Application of A^n and eigenvectors. This example **will not be tested**.
Consider a company with 1000 machines.
a) Suppose a working machine has a 99% probability of working tomorrow. Suppose a broken machine has a 50% probability of being broken tomorrow. Write down the probability matrix, A.

$$A = \begin{array}{c} \\ W \\ B \end{array}\begin{array}{c} \begin{array}{cc} W & B \end{array} \\ \begin{bmatrix} 0.99 & 0.5 \\ 0.01 & 0.5 \end{bmatrix} \end{array} \leftarrow \text{today}$$

tomorrow

b) Suppose all machines are working today. Write down the initial state vector, $\vec{v}$.

$$\vec{v} = \begin{array}{c} W \\ B \end{array}\begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

c) How many machines will be working or broken tomorrow?

$$A\vec{v} = \begin{array}{c} W \\ B \end{array}\begin{bmatrix} 990 \\ 10 \end{bmatrix}$$

d) How many machines will be working or broken two days from now?

$$A^2\vec{v} \approx \begin{bmatrix} 985 \\ 15 \end{bmatrix}$$

e) How many machines will be working or broken three days from now?

$$A^3\vec{v} \approx \begin{bmatrix} 983 \\ 17 \end{bmatrix}$$

f) How many machines will be working or broken n days from now, where n is a non-negative integer?

$A^n\vec{v}$ Note: $\lim_{n\to\infty} A^n\vec{v} = \begin{bmatrix} \frac{50000}{51} \\ \frac{1000}{51} \end{bmatrix} \approx \begin{bmatrix} 980 \\ 20 \end{bmatrix}$

g) What initial state vector $\vec{v}$ would have $A\vec{v} = \vec{v}$? This is called the **steady-state vector** because the state after one day is the same as the initial state.

$\begin{bmatrix} 980 \\ 20 \end{bmatrix}$ or any eigenvector of A corresponding to $\lambda = 1$