Example Continued...

$$\lambda^3 - 12\lambda + 16 = 0$$

Possible roots: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

$\lambda = -1$: $(-1)^3 - 12(-1) + 16 = 0$? No

$\lambda = 1$: $1^3 - 12(1) + 16 = 0$? No

$\lambda = -2$: $(-2)^3 - 12(-2) + 16 = 0$? No

$\lambda = 2$: $2^3 - 12(2) + 16 = 0$? YES

$\lambda = 2$ is a root

$\Rightarrow$ $\lambda - 2$ is a factor of polynomial

$$
\begin{array}{r l}
 & \lambda^2 + 2\lambda - 8 \\
\cline{2-2}
(\lambda - 2) \big) & \lambda^3 + 0\lambda^2 - 12\lambda + 16 \\
 & -(\lambda^3 - 2\lambda^2) \\
\cline{2-2}
 & 2\lambda^2 - 12\lambda + 16 \\
 & -(2\lambda^2 - 4\lambda) \\
\cline{2-2}
 & -8\lambda + 16 \\
 & -(-8\lambda + 16) \\
\cline{2-2}
 & 0
\end{array}
$$

$$\lambda^3 - 12\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda^2 + 2\lambda - 8) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda + 4) = 0$$

$$(\lambda - 2)^2(\lambda + 4) = 0$$

$$\lambda = 2, -4$$

$$\lambda = 2, 2, -4$$

Definition: The **characteristic equation** of A is $|A - \lambda I| = 0$.
The **algebraic multiplicity** of an eigenvalue λ_i is the exponent on $(\lambda_i - \lambda)$ in the characteristic equation. or $(\lambda - \lambda_i)$
The **geometric multiplicity** of an eigenvalue is the number of basis vectors in the corresponding eigenspace.

Example: Let A have characteristic equation $(7-\lambda)^3(9-\lambda)^2 = 0$. A basis for E_7 consists of one vector. A basis for E_9 consists of two vectors. Find the eigenvalues of A and state their algebraic multiplicities and their geometric multiplicities.

Eigenvalues	Alg. Mult.	Geo. Mult.
$\lambda = 7$	3	1
$\lambda = 9$	2	2

eigenspace is a plane
$\vec{x} = [\ ]s + [\ ]t$

eigenspace is a line
$\vec{x} = [\ ]t$

Fact: For each eigenvalue:
$1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

Comment: If a matrix has
(geometric multiplicity) = (algebraic multiplicity) for all its eigenvalues
then the matrix has a nice property.
We'll see the details in Section 4.4.

We'll look at five properties of eigenvalues.

Property 1: A is invertible if and only if 0 is not an eigenvalue of A.

Example: Prove Property 1.

0 is <u>not</u> an eigenvalue of A

$\iff$ $\det(A - 0I) \neq 0$

$\iff$ $\det A \neq 0$

$\iff$ A is invertible

Property 2: If A is invertible and $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

Example: Prove Property 2.

$A\vec{x} = \lambda\vec{x}$ and A^{-1} exists

$$\Rightarrow \underbrace{A^{-1}A}\vec{x} = A^{-1}\lambda\vec{x}$$

$$\Rightarrow \cancel{I}\vec{x} = \lambda A^{-1}\vec{x}$$

$$\Rightarrow \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x} \quad (\lambda \neq 0 \text{ by Property 1})$$

$$\Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$

eigenvector of A^{-1}

eigenvalue of A^{-1}

Property 3: Let n be a non-negative integer. If $A\vec{x} = \lambda\vec{x}$ then $A^n\vec{x} = \lambda^n\vec{x}$.

Example: Prove Property 3.

$$A^n\vec{x} = A^{n-1}A\vec{x}$$

$$= A^{n-1}\lambda\vec{x}$$

$$= \lambda A^{n-1}\vec{x}$$

$$= \lambda A^{n-2}A\vec{x}$$

$$\vdots$$

$$= \lambda^n\vec{x}$$

eigenvalue of A^n

eigenvector of A^n

Property 4: If $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of $A + kI$ with eigenvalue $\lambda + k$.

Example: Prove Property 4.

$$(A+kI)\vec{x} = A\vec{x} + kI\vec{x}$$
$$= \lambda\vec{x} + k\vec{x}$$
$$= \underbrace{(\lambda+k)}_{\text{eigenvalue of } A+kI}\underbrace{\vec{x}}_{\text{eigenvector of } A+kI}$$

Property 5: Let n be a non-negative integer.
Suppose A has eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then:
$A^n(c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_m\vec{x}_m) = c_1\lambda_1^n\vec{x}_1 + c_2\lambda_2^n\vec{x}_2 + \ldots + c_m\lambda_m^n\vec{x}_m$.

Comment: This is a generalization of Property 3.
Note that the coefficients are preserved.

Example: Suppose A has:
eigenvalue $\lambda_1 = -2$ corresponding to eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and
eigenvalue $\lambda_2 = 3$ corresponding to eigenvector $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Calculate $A^3 \begin{bmatrix} 11 \\ 2 \end{bmatrix}$.

(Property 5)

1) Let $c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & -1 & 2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right] \quad \text{RREF}$$

$c_1 = 3, \; c_2 = 4$

2) $$A^3 \begin{bmatrix} 11 \\ 2 \end{bmatrix} = A^3 (3\vec{v}_1 + 4\vec{v}_2)$$
$$= c_1\lambda_1^n \vec{v}_1 + c_2\lambda_2^n \vec{v}_2$$
$$= c_1\lambda_1^3 \vec{v}_1 + c_2\lambda_2^3 \vec{v}_2$$
$$= \underbrace{\cancel{3(-2)^3}}_{-24}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underbrace{\cancel{4(3^3)}}_{108}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 192 \\ -156 \end{bmatrix}$$

Example: Suppose A has the eigenvalue 3 corresponding to the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

List one eigenvector and one eigenvalue for each of the following matrices: $A^{-1}, A^4, A + 2I$.

(Properties 2-4)

Matrix	Eigenvector	Eigenvalue
A^{-1}	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$\frac{1}{3}$
A^4	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$3^4 = 81$
$A+2I$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$3+2=5$

4.4 Diagonalization

Definition: An $n \times n$ matrix A is **diagonalizable** if there exist an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

Fact: To find P we find a basis for each eigenspace of A. The basis vectors go into the columns of P. The matrix D has the eigenvalues on the diagonal, in the same order as P.

Example: Let $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find P and D that diagonalize A.

$\lambda = 2, 3, 3$
(A is upper triangular)

E_2: set of all eigenvectors corresponding to $\lambda = 2$, plus the zero vector.

$$[A - \lambda I \mid \vec{0}]$$

$$[A - 2I \mid \vec{0}]$$

$$\vdots$$

$$[\quad \mid \quad] \text{ RREF}$$

$$\vec{x} = [\quad] t$$

Basis for $E_2 = \{[\quad]\}$